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On the torsion part in the K -theory of imaginary quadratic fields

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Abstract. We obtain upper bounds for the torsion in the K -groups of the ring of integers of imaginary quadratic number fields, in terms of their discriminants.

1. Introduction

Let F be a number field with ring of integers \mathcal{O}_F , and denote by $K_n(\mathcal{O}_F)$ Quillen's K -theory group of degree n . By a theorem of Quillen $K_n(\mathcal{O}_F)$ is finitely generated, and its rank was computed by Borel for each $n \geq 2$ [4]. We refer to [13] for a detailed survey concerning the problem of computing these groups. In this paper we are interested in the problem of finding upper bounds for the torsion part of $K_n(\mathcal{O}_F)$. We obtain a result in the case of imaginary quadratic fields:

Theorem 1.1. *Let $n \geq 2$. There exists a constant $C(n)$ such that for every imaginary quadratic number field F , the group $K_n(\mathcal{O}_F)$ contains no p -torsion element for any prime p with*

$$\log(p) > C(n)|D_F|^{2n(n+1)}, \quad (1.1)$$

where D_F denotes the discriminant of F .

For $n = 2$ the better estimate

$$\log |K_2(\mathcal{O}_F) \otimes \mathbb{Z}[1/6]| \leq C \cdot |D_F|^2 \log |D_F| \quad (1.2)$$

was obtained in [6]. Tensoring by $\mathbb{Z}[1/6]$ excludes the 2- and 3-torsion in the bound. In a similar way, Theorem 1.1 is obtained from an upper bound for $|\text{tors } K_n(\mathcal{O}_F)|$ that holds modulo small torsion, although we need to exclude more primes here (see Proposition 4.1).

Let us briefly indicate the strategy. It is classical that $K_n(\mathcal{O}_F)$ relates directly to the homology of $\text{GL}_N(\mathcal{O}_F)$, for $N = 2n + 1$ (see Section 2). The general idea for

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the proof of Theorem 1.1 is to obtain an upper bound for the homology of $GL_N(\mathcal{O}_F)$ by using its action on the symmetric space $X = PGL_N(\mathbb{C})/PU(N)$. A theorem of Gelander (Theorem 3.2) controls the topology of noncompact arithmetic quotients of X in terms of their volume, and from this we can obtain an upper bound for the torsion homology of $PGL_N(\mathcal{O}_F)$ (Section 3). A spectral sequence argument finally provides the bound for $\text{tors } H_n(GL_N(\mathcal{O}_F))$. Unfortunately the constant $C(n)$ in (1.1) is not explicit, and its appearance is explained by some nonexplicit constant in Gelander’s theorem.

Our result can be compared with the upper bounds obtained by Soulé in [11, Prop.4 iii)]. He showed that for F imaginary quadratic one has “up to small torsion”:

$$\log |\text{tors } K_n(\mathcal{O}_F)| \leq |D_F|^{1120n^4 \log(n)}. \tag{1.3}$$

Thus, Theorem 1.1 improves (asymptotically) the upper bound for the existence of p -torsion in $K_n(\mathcal{O}_F)$ that follows from (1.3). Soulé’s work also contains general upper bounds – without restriction on the signature of the number field F – which have been later improved in [2]. It is not clear if the strategy for Theorem 1.1 could be adapted to include the case of number fields with more than one archimedean places (see Remark 4.3). Until Section 4 and the last step in the proof, we will be treating the case of general number fields F .

A main motivation for the study of upper bounds in K -theory comes from the case $F = \mathbb{Q}$, for which the triviality of $K_{4m}(\mathbb{Z})$ is known to be equivalent to the Vandiver conjecture (cf. [10]). To our knowledge, the best available bounds (up to small torsion) for these groups are given in [11, Prop. 4 iv)]. The method for Theorem 1.1 applies directly to the case $F = \mathbb{Q}$, and an effective version of Gelander’s theorem would thus provide an upper bound for $K_n(\mathbb{Z})$. However, it might well be that the bounds obtained this way will be larger than Soulé’s bounds.

2. Preliminaries

2.1 For an abelian group A we denote by $\text{tors } A$ its torsion subgroup. For $\ell > 1$, we define

$$\text{tors}_\ell A = \text{tors} (A \otimes \mathbb{Z}[1/m]), \tag{2.1}$$

where m is the lowest common multiple to all integers $\leq \ell$. Thus, tors_ℓ kills all torsion up to ℓ .

2.2 Let Γ be a group. The symbol $H_n(\Gamma)$ will denote the homology of Γ with coefficients in \mathbb{Z} . Since $\mathbb{Z}[1/m]$ is a flat \mathbb{Z} -module, it follows from the universal coefficient theorem [5, p. 8] that

$$\text{tors}_\ell H_n(\Gamma) \cong H_n(\Gamma, \mathbb{Z}[1/m]), \tag{2.2}$$

for m as above. In this context, we will also need the following fact.

Lemma 2.1. *Let $\Gamma_0 \subset \Gamma$ be a normal subgroup of index $[\Gamma : \Gamma_0] \leq \ell < \infty$, and m be the lowest multiple common to all integers $\leq \ell$. Then $H_n(\Gamma, \mathbb{Z}[1/m])$ injects into $H_n(\Gamma_0, \mathbb{Z}[1/m])$.*

Proof. Let $A = \Gamma/\Gamma_0$. Since $|A| = [\Gamma : \Gamma_0]$ is invertible in $\mathbb{Z}[1/m]$, the transfer map (cf. [5, III.9–10]) gives an isomorphism between $H_n(\Gamma, \mathbb{Z}[1/m])$ and the module of co-invariants $H_n(\Gamma_0, \mathbb{Z}[1/m])_A$. But for $|A|$ invertible, the norm map (cf. [5, III.1]) provides an isomorphism of the latter with the submodule $H_n(\Gamma_0, \mathbb{Z}[1/m])^A$ of A -invariant elements in $H_n(\Gamma_0, \mathbb{Z}[1/m])$. \square

2.3 Let F denote a number field of degree d . Until Section 4 there is no restriction on d .

The group $K_n(\mathcal{O}_F)$ is defined as the n -th homotopy group $\pi_n(BGL(\mathcal{O}_F)^+)$ of Quillen’s plus construction applied on the classifying space of the linear group $GL(\mathcal{O}_F) = \varinjlim GL_N(\mathcal{O}_F)$. Since the (integral) homology of $BGL(\mathcal{O}_F)^+$ is canonically isomorphic to the homology of $GL(\mathcal{O}_F)$, the Hurewicz map provides a homomorphism

$$K_n(\mathcal{O}_F) \rightarrow H_n(GL(\mathcal{O}_F)). \tag{2.3}$$

The kernel of the Hurewicz map (2.3) does not contain p -torsion elements, for any $p > \frac{n+1}{2}$ (see [1]). It follows that for $\ell \geq \frac{n+1}{2}$, the group $\text{tors}_\ell K_n(\mathcal{O}_F)$ injects into $\text{tors}_\ell H_n(GL(\mathcal{O}_F))$. Moreover, the stability result of Maazen (see van der Kallen [12, Theorem 4.11]) tells us that the homology group $H_n(GL(\mathcal{O}_F))$ equals $H_n(GL_N(\mathcal{O}_F))$ for any $N \geq 2n + 1$. These various results provide the next proposition, which is already a basic ingredient in Soulé’s method [11].

Proposition 2.2. *For $N \geq 2n + 1$ and $\ell \geq \frac{n+1}{2}$, there is an injective map*

$$\text{tors}_\ell K_n(\mathcal{O}_F) \rightarrow \text{tors}_\ell H_n(GL_N(\mathcal{O}_F)).$$

3. Torsion homology of $PGL_N(\mathcal{O}_F)$

3.1 Consider the semisimple Lie group $G = PGL_N(F \otimes_{\mathbb{Q}} \mathbb{R})$. The arithmetic subgroup $\Gamma = PGL_N(\mathcal{O}_F)$ is a nonuniform (irreducible) lattice in G . The following fact is well-known (see for instance [8, Lemma 13.1]):

Lemma 3.1. *There exist a constant $\gamma = \gamma(d, N)$ such that for any number field F of degree d the group $\Gamma = PGL_N(\mathcal{O}_F)$ has a torsion-free normal subgroup Γ_0 with index $[\Gamma : \Gamma_0] \leq \gamma$.*

The result essentially follows from Minkowski’s lemma that asserts that the reduction map $GL_n(\mathbb{Z}) \rightarrow GL(\mathbb{Z}/m)$ has torsion-free kernel for $m > 2$. This permits to give an explicit value for the bound γ . See [3, Sect. 2] for an modified argument that provides better (i.e., lower) value for this constant.

3.2 Let X be the symmetric space associated with G , i.e, $X = G/K$ for some maximal compact subgroup $K \subset G$. Note that X depends only on N and the signature of F . For Γ_0 as in Lemma 3.1 the quotient $M = \Gamma_0 \backslash X$ is a noncompact

locally symmetric space. Gelander’s theorem [8, Theorem 1.5 (1)] shows that M is homotopy equivalent to a simplicial complex of bounded size. Let us define a (δ, v) -simplicial complex to be a simplicial complex with at most v vertices, all of valence at most δ . Then the precise result is the following.

Theorem 3.2. (Gelander)

There exist constants $\delta = \delta(X)$ and $\alpha = \alpha(X)$ such that any noncompact arithmetic manifold $M = \Gamma_0 \backslash X$ is homotopically equivalent to a $(\delta, \alpha \text{vol}(M))$ -simplicial complex \mathcal{K} .

Remark 3.3. It is clear that the result holds for any normalization of the volume vol on X , by adapting the constant α accordingly.

Remark 3.4. The similar result for compact manifolds has been recently obtained in [7].

A result of Gabber allows to bound torsion homology of a simplicial complex. We will use this result in the following form.

Proposition 3.5. Let \mathcal{K} be a (δ, v) -simplicial complex. Then the torsion homology of \mathcal{K} is bounded as follows:

$$\begin{aligned} \log |\text{tors } H_n(\mathcal{K})| &\leq v \cdot \binom{\delta}{n} \\ &\leq v \cdot \delta^n \end{aligned} \tag{3.1}$$

Proof. The number of n -simplices in \mathcal{K} is bounded above by $\frac{v}{n+1} \binom{\delta}{n}$. By Gabber’s lemma (cf. [6, Lemma 2.2]), since the chain complex \mathcal{K}_\bullet is a simplicial complex we may bound the torsion homology as follows:

$$|\text{tors } H_n(\mathcal{K}_\bullet)| \leq \sqrt{n+2}^{\text{rank}(\mathcal{K}_n)},$$

from which the result follows. □

3.3 From Theorem 3.2 and Proposition 3.5 we obtain

$$\begin{aligned} \log |\text{tors } H_n(\Gamma_0)| &\leq \alpha \delta^n \text{vol}(\Gamma_0 \backslash X) \\ &\leq \alpha \delta^n \gamma \text{vol}(\Gamma \backslash X), \end{aligned} \tag{3.2}$$

for constants α, γ and δ that can be chosen to depend only on N and d .

Proposition 3.6. Let $N = 2n + 1$ and $\Gamma = \text{PGL}_N(\mathcal{O}_F)$. For some constant $C(d, n)$, we have the following upper bound for the torsion homology:

$$\log |\text{tors}_\gamma H_n(\Gamma)| \leq C(d, n) |D_F|^{2n(n+1)}, \tag{3.3}$$

where $\gamma = \gamma(d, 2n + 1)$ is the constant in Lemma 3.1.

Proof. By Section 2.2 and Lemma 2.1, it is enough to bound

$$\begin{aligned} |\text{tors } H_n(\Gamma_0, \mathbb{Z}[1/m])| &= |\text{tors } H_n(\Gamma_0) \otimes \mathbb{Z}[1/m]| \\ &\leq |\text{tors } H_n(\Gamma_0)|. \end{aligned}$$

Using (3.2), it remains to bound the covolume of $\Gamma = \text{PGL}_N(\mathcal{O}_F)$. It is certainly bounded by the covolume of $\text{PSL}_N(\mathcal{O}_F)$. We can choose (cf. Remark 3.3) to work with the normalization of the volume used in Prasad’s volume formula: let us assume that

$$\text{vol}(\text{PSL}_N(\mathcal{O}_F)\backslash X) = \mu(\text{SL}_N(\mathcal{O}_F)\backslash \tilde{G}), \tag{3.4}$$

where $\mu = \mu_\infty = \mu_S$ is the Haar measure on $\tilde{G} = \text{SL}_N(F \otimes_{\mathbb{Q}} \mathbb{R})$ defined in [9, 3.6]. Then the volume formula [9, Theorem 3.7] shows that

$$\text{vol}(\text{PSL}_N(\mathcal{O}_F)\backslash X) = A \cdot |D_F|^{\frac{N^2-1}{2}} \prod_{k=2}^N \zeta_F(k),$$

for some (explicit) constant A that depends only on N and the signature of F . The product of zeta functions is easily shown to be $\leq 2^{N-1}$, so that the result follows. \square

Remark 3.7. Note that if we had explicit constants α and δ in Theorem 3.2 (for some explicit normalization of the volume), the procedure would permit to make the constant $C(d, n)$ explicit.

4. Upper bound in the K -groups

We finally state and prove the following result, of which Theorem 1.1 will be an immediate consequence (see proof below). Recall that the notation tors_γ was introduced in Section 2.1.

Proposition 4.1. *Let $n \geq 2$. There are constants $C(n)$ and $\gamma = \gamma(n)$ such that for any imaginary quadratic field F , one has*

$$\log |\text{tors}_\gamma K_n(\mathcal{O}_F)| \leq C(n) |D_F|^{2n(n+1)}. \tag{4.1}$$

Proof. With γ as in Lemma 3.1 and larger than $\frac{n+1}{2}$, we have from Proposition 2.2:

$$\begin{aligned} |\text{tors}_\gamma K_n(\mathcal{O}_F)| &\leq |\text{tors}_\gamma H_n(\text{GL}_N(\mathcal{O}_F))| \\ &= |H_n(\text{GL}_N(\mathcal{O}_F), \mathbb{Z}[1/m])|, \end{aligned} \tag{4.2}$$

with $N = 2n + 1$ and m the lowest multiple common to all integers $\leq \gamma$.

With $\Gamma = \text{PGL}_N(\mathcal{O}_F)$, we have the short exact sequence

$$1 \rightarrow \mathcal{O}_F^\times \rightarrow \text{GL}_N(\mathcal{O}_F) \rightarrow \Gamma \rightarrow 1, \tag{4.3}$$

and for it the Lyndon-Hochschild-Serre spectral sequence (cf. [5, VII.6]) reads:

$$E_{pq}^2 = H_p(\Gamma, H_q(\mathcal{O}_F^\times, \mathbb{Z}[1/m])) \implies H_{p+q}(\mathrm{GL}_N(\mathcal{O}_F), \mathbb{Z}[1/m]). \quad (4.4)$$

Since we assume that F is imaginary quadratic, we have that \mathcal{O}_F^\times is finite, of order ≤ 6 . Then $H_q(\mathcal{O}_F^\times, \mathbb{Z}[1/m]) = 0$ for $q > 0$, and the spectral sequence is concentrated in $q = 0$. Note also that since the image of \mathcal{O}_F^\times is central in $\mathrm{GL}_N(\mathcal{O}_F)$, the action of Γ on $H_0(\mathcal{O}_F^\times, \mathbb{Z}[1/m]) = \mathbb{Z}[1/m]$ is trivial (cf. [5, ex. 1 p.80]). We obtain:

$$H_n(\Gamma, \mathbb{Z}[1/m]) = H_n(\mathrm{GL}_N(\mathcal{O}_F), \mathbb{Z}[1/m]),$$

and the result follows from Proposition 3.6 with the same constant $C(n) = C(2, n)$.
□

Remark 4.2. As it follows from the discussion in Section 3.1, the constant γ can be easily made explicit – contrarily to $C(n)$.

Remark 4.3. For F with more than one archimedean place, the spectral sequence (4.4) does not collapse at E_{pq}^2 . A priori the differential ∂_r (with $r \geq 2$) might add torsion, whenever both its domain and image is infinite. It seems difficult to control this phenomena to obtain upper bounds in this more general setting.

Proof of Theorem 1.1. In Proposition 4.1, we can assume that $C(n) \geq \log(\gamma)$ (by increasing the constant if necessary). Let $s \in K_n(\mathcal{O}_F)$ be an element of order p , with p prime such that $\log(p) > C(n)|D_F|^{2n(n+1)}$. In particular, we have $p > \gamma$, so that the image of s in $\mathrm{tors}_\gamma K_n(\mathcal{O}_F)$ is non-trivial. But this contradicts the bound (4.1).

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Declarations

Data availability This work is purely based on a theoretical approach; no datasets were used.

Conflict of interest The author states that there is no conflict of interest.

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