Arbitrage, the limit order book and market microstructure aspects in financial market models

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Arbitrage, the limit order book and market microstructure aspects in financial market models

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Kurzfassung

Diese Arbeit gliedert sich in drei Teile.

Der erste Teil trägt den Titel "Arbitragemöglichkeiten nach dem Wechsel zu einem nicht-äquivalenten Mass". Hier untersuchen wir Arbitragemöglichkeiten in diversifizierten Märkten wie sie von Robert Fernholz im Jahre 1999 [Fer99] eingeführt wurden. Mit Hilfe einer bestimmten Masswechseltechnik können wir eine Vielzahl von diversifizierten Märkten konstruieren. Die Konstruktion basiert auf einem Masswechsel, der zu einem absolut stetigen, aber nicht äquivalenten Mass bezüglich des Ausgangsmasses führt. Damit diese Technik funktioniert, setzen wir eine wichtige Nichtdegeniertheitsbedingung voraus. Ausserdem gehen wir auf die Dynamik des Preisprozesses unter dem neuen Mass ein und geben einen Ausblick auf weitere Anwendungen im Rahmen von Währungs- und Anleihemärkten.
Teile dieses ersten Abschnittes wurden in der Zeitschrift *Annals of Finance* unter dem Titel "Arbitrage opportunities in diverse markets via a non-equivalent change of measure" [OR06] veröffentlicht.

Im zweiten Teil betrachten wir ein Modell für das Limitorderbuch.
Hier beschäftigen wir uns damit, wie man das Orderbuch beschreiben kann. Insbesondere wollen wir darstellen, wie Order ankommen, wie man sie speichern kann und wie sie entweder ausgeführt oder gelöscht werden.
Wir schlagen das Konzept der zufälligen Masse vor, um das Limitorderbuch zu modellieren. Mit diesem Hilfsmittel lassen sich die Ankunftszeit, die Größe, der relative Preis sowie der absolute Preis eines Limitorders speichern. Die Eingabeparameter unseres Modells können an empirische Beobachtungen angepasst werden. Es lässt sich zeigen, dass unter gewissen Annahmen an den Orderankunftsprozess das Limitorderbuch zu jedem Zeitpunkt die Differenz zweier doppelt stochastischer Poissonprozesse ist.
Sobald man das Orderbuch kennt, kann man Geld- und Briefkurse definieren. Damit lassen sich Eigenschaften der Geld-Brief Spanne untersuchen. Wir zeigen, dass unter bestimmten Bedingungen die Verteilung der Preise im Orderbuch immer noch langschwänzig ist, falls die Ankunftsverteilung der relativen Preise diese Eigenschaft hat und geben einen Ausdruck für das durchschnittliche Orderbuch auf lange Sicht. Das Löschen von Ordern wird anhand der Warteschlangentheorie behandelt.
In besonderen Fällen kann die Wartezeit bis zur Ausführung oder Streichung explizit berechnet werden. Die wichtigen Konzepte Zeithorizont und Ausführungswahrscheinlichkeit werden definiert und behandelt. Unser Ansatz gibt uns die Möglichkeit beide Verteilungen zu berechnen und einige ihrer Eigenschaften herzuleiten. Schliesslich betrach-ten wir auch die Erweiterung auf ein Grossinvestormodell. Wir geben Bedingungen an unter denen dieses erweiterte Modell immer noch arbitragefrei bleibt.

Der dritte Teil beschreibt ein dynamisches Marktmikrostrukturmodell, in dem ein strategischer Market Maker mit einem informierten Händler konkurriert. Wir betrachten zusätzlich einen sogenannten "noise trader" sowie Händler, die Limitorder aufgeben. Unser Modell ist ein Mehrperiodenmodell. Wir stellen notwendige und hinreichende Bedingungen dafür auf, dass ein Gleichgewicht in unserem Markt herrscht. Zusätzlich versuchen sowohl der informierte Händler als auch der Market Maker ihren Gewinn zu maximieren. Die sich ergebenden rekursiven Gleichungen führen zu vielfältigen wirtschaftlichen Interpretationsmöglichkeiten. Insbesondere betrachten wir auch das Zusammenspiel verschiedener Arten von Information. Auch gehen wir auf den Fall ein, dass es mehrere Market Maker gibt. Unser Modell ist so allgemein, dass es uns erlaubt, einige der bekannten Marktmikrostrukturmodelle auf einfache Art und Weise herzuleiten, darunter die exzellenten Arbeiten von Kyle [Kyl85] und Bondarenko und Sung [BJ03] sowie einige bekannte Ergebnisse in diesem Bereich, wie z.B. die Erkenntnis, dass kein Gleichgewicht existiert, falls wir nur einen gewinnmaximierenden Market Maker betrachten und das Orderbuch weglassen, wie z.B. in Dennert [Den93] oder Bondarenko [Bon01].

Abstract

This dissertation consists of three parts.

The first is entitled "Arbitrage opportunities after a non-equivalent change of measure". Here we study arbitrage opportunities in diverse markets as introduced by Robert Fernholz in 1999 [Fer99]. By a change of measure technique we are able to generate a variety of diverse markets. The construction is based on an absolutely continuous but non-equivalent measure change which implies the existence of instantaneous arbitrage opportunities in diverse markets. For this technique to work, we single out a crucial non-degeneracy condition. Moreover, we discuss the dynamics of the price process under the new measure as well as further applications.
Parts of this first section were published in *Annals of Finance* under the title "Arbitrage opportunities in diverse markets via a non-equivalent change of measure" [OR06].

In the second part, we look at a model for the limit order book.
Here we deal with the issue of how to construct a framework for order arrival, storage, cancellation and execution. We propose to use random measures to describe the limit order book. With this tool, we can store the arrival time, the size, the relative price as well as the absolute price of each limit order. Our model is flexible enough to be fitted to empirical observations of the limit order arrival process. Based on this model of the order book, a large variety of observations can be considered. It turns out that, based on certain assumptions on the order arrival process, the limit order book will be the difference of two doubly stochastic Poisson processes at every point in time. Endogenously, bid and ask prices arise and we investigate properties of the bid-ask spread. Naturally a new type of options occurs, a reverse Asian fixed strike lookback option. We show that under certain conditions the distribution of prices in the order book is still heavy-tailed if the arrival process has the same property and give an expression for the average order book in the long-run. Cancellation of orders is included using ideas from queuing theory. In particular cases, the waiting time until cancellation and execution can be calculated explicitly. We introduce the important concepts of time horizons and execution probabilities. Our approach gives us the possibility to calculate both distributions and derive some of their properties. Finally, we also look at an extension to a large trader model. We give conditions under which this extended model will still be arbitrage-free.

Finally, the third part deals with a dynamic market microstructure model, in which a strategic market maker competes with an informed trader. We include the presence of noise traders and limit order traders in our setup. Our model is a $N-$period model. We give necessary and sufficient conditions for an equilibrium to exist and provide conditions for it to be unique. Moreover, both the informed trader and the market maker try to maximize their profits.

The resulting recursive equations lead to various economic interpretations. We investigate the interplay of different information sets. Finally we consider the competitive situation for the market maker. Our framework is general enough to obtain several well-known models in a straightforward way, among them the excellent models by Kyle [Kyl85] and by Bondarenko and Sung [BJ03] as well as certain well-known results such as the non-existence of an equilibrium if there is no order book and only one competitive market maker, see e.g. Dennert [Den93] or Bondarenko [Bon01].

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Introduction

"Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry."

- Bertrand Russell, in Study of Mathematics, about the beauty of Mathematics

Diverse markets and non-equivalent measure changes

The first revolution in finance began in 1952 with the publication of "Portfolio Selection", an early version of the doctoral dissertation of Harry Markowitz. This lead to a shift away from trying to find the best stock for an investor towards the concept of understanding the trade-off between risk and return in a portfolio of stocks. William Sharpe helped in implementing the ideas of Markowitz by introducing the concept of looking at covariances not only between every possible pair of stocks but also between each stock and the market. In that framework one can also address the issue of optimization. Keeping the risk of the portfolio below a specified acceptable threshold one tries to maximize the mean rate of return. Subsequently, in 1990, Markowitz, Sharpe and Miller received the first Nobel price in economics ever awarded for work in finance. Over time, the mathematics involved in those ideas has become increasingly sophisticated.

Already in 1944, Kiyosi Itô published his first paper on stochastic integration. He tried to construct a true stochastic differential which could be used when investigating Markov processes. We use his own words to describe this "In these papers [Kolmogorov (1931) and Feller (1936)] I saw a powerful analytic method to study the transition probabilities of the process, namely Kolmogorov's parabolic equation and its extension by Feller. But I wanted to study the paths of Markov processes in the same way as Lévy observed differential processes. Observing the intuitive background in which Kolmogorov derived his equation, I noticed that a Markovian particle would perform a time homogeneous differential process for infinitesimal future at every instant, and arrived at the notion of stochastic differential equation governing the paths of a Markov process that could be formulated in terms of the differentials of a single differential process." (Itô, 1987). 1951 marked the year when he published his celebrated Itô's formula. In the subsequent years, the theory of stochastic integration was greatly extended, notably by Doob in 1953 to processes with conditionally orthogonal increments. Samuelson introduced a new model of the stock price in 1965, which is now known as geometric Brownian motion thus leaving the arithmetic Brownian motion world of Bachelier (1900). In particular, during the past three decades, pricing and hedging of derivative securities as well as portfolio optimization have gained considerable interest in both the financial and the academic world.

The second revolution in finance, which also led to the breakthrough of mathematical applications took place in the early seventies with the seminal papers of Black and Scholes (1973) and Merton (1969, 1971, 1973, 1977). In 1997, Merton and Scholes were awarded a Nobel prize in economics for pricing European call and put options and the underlying economic reasoning. More precisely, within Samuelson's framework, they used geometric Brownian motion as a model for the stock price and priced European options on stocks by the initial capital that is required to replicate the pay-off of the option at maturity by continuously trading in the underlying stock and a riskless asset. The replicating strategy has to be self-financing in the sense that after an initial investment neither additional funds are invested nor extra funds are withdrawn. Under no arbitrage, the price of the option must then equal the value of the replicating portfolio at any point in time.

One important branch in mathematical finance is portfolio optimization. Merton applied results from stochastic control theory to consider a framework, in which investors are allowed to trade dynamically in the security market and thus can maximize their expected utility both from consumption and wealth during some prespecified period of time. In a Markovian setting, this approach leads to solving Hamilton-Jacobi-Bellman equations of dynamic programming and thus typically to nonlinear partial differential equations. However, once we leave the world of utility functions with hyperbolic absolute risk aversion, we usually have to use numerical procedures to find solutions.

A second approach, which is now at the core of mathematical finance, was then developed; the martingale approach for pricing and hedging derivatives and for Merton's portfolio optimization problem. In 1976, Cox and Ross found out that the price of an option in a complete market situation must be determined as if investors were risk-neutral, meaning that they had probability beliefs such that the riskless interest rate is equal to the stock's expected rate of return.

Accordingly, the price of a replicable option can be computed as the expected value of the discounted pay-off under the risk-neutral probability measure. Harrison and Kreps (1979), Kreps (1981) and Harrison and Pliska (1981) formalized this idea and worked out the connection between the concepts of no-arbitrage, equivalent local martingale measures and the pricing of replicable derivatives. Intuitively we redistribute the probability mass such that the increments of the discounted stock price will be zero in expected values without changing the set of events that receive positive probability. Formally, it is a probability measure equivalent to the real-world probability and such that the discounted price of the stock is a local martingale. Some effects which occur if the change of measure is only absolutely continuous but not necessarily equivalent will be treated in the first chapter.

The basis of mathematical finance then rests on the central result that absence of arbitrage is essentially equivalent to the existence of an equivalent local martingale measure. In discrete time, it was proved by Dalang, Morton and Willinger (1990) that the word essentially can be deleted. Back and Pliska (1991), Schachermayer (1992), Kabanov and Kramkov (1994) and Rogers (1995) provided alternative and somewhat simpler proofs of the same result. When going over to continuous time, things become more complicated and we usually need stronger assumptions than just no arbitrage. In its full generality, conditions for the existence of equivalent local martingale measures can be found in Delbaen and Schachermayer (1994, 1997) and the references therein.

In recent years, models for pricing and hedging of derivatives and for portfolio optimization problems have been generalized and refined to become increasingly realistic. There is now a large list of publications and we consequently refer to the books by Duffie (1992), Lamberton and Lapeyre (1996), Karatzas (1997), Musiela and Rutkowski (1997), Karatzas and Shreve (1998) and Korn (1998).

One also tried to extend the model of the stock price process to include various phenomena which exist in the real markets. Among others, researchers have been looking at transaction costs, at effects with different sets of information and at large traders.

Here we look at another restriction on the financial market. We impose a condition, which basically means that no single stock is ever allowed to dominate the entire market. This is a feature which one would like to have in any model of the financial markets because we observe that phenomenon in real markets.

Therefore, it is worthwhile investigating how this additional feature relates to existing models of the market, how we can make it compatible with them and whether the new market is arbitrage-free.

The limit order book and its applications

In recent years we can observe that more and more of the major exchanges in the world rely upon limit orders for the provision of liquidity. Data has only been available for the last few years and some empirical investigations have been undertaken in the literature, see among many others Bouchaud et al. (2002) and Bouchaud and Potters (2003).

Electronic trading with the help of a public limit order book continues to increase its share in worldwide security trading. In order-driven markets investors can submit either market or limit orders.

Limit orders are stored in the book of the exchange and executed according to different rules which vary from exchange to exchange.

The advantage of using limit orders is that, by delaying transactions, patient traders may be able to trade at a more favorable price. However we have to deal with the uncertainty as to when the orders are executed and if they are executed at all. In addition, the order may be cancelled. This can be caused by the trader's perception about the fair price of an asset, which may have changed since the time the order was placed.

An excellent overview of the literature on order books is given in Smith et al. (2003). Here we summarize those findings. In 1985, Bias et al. were one of the first that looked into the real-world limit order markets. They conduct a comprehensive empirical study of the order flow and the limit order book at the Paris Bourse and develop a theoretical model of limit order submissions. Different types of traders exhibit different behaviour in the limit order markets. Among others, Lo et al. (2002) and Hollifield et al. (2004) try to explain the rationale behind this. Many more authors have done research on the microstructure of double auction markets, limit order books, order flow, traded volume and the bid-ask spread.

There are two independent lines of prior work, one in the financial economics literature and the other in the physics literature. In the economics literature, we mainly deal with a static order process and base the models on econometrics. On the other hand, the models in the physics literature are mostly artificial toy models, but they are fully dynamic since they allow the order process to react to changes in prices. Random order placement with periodic clearing was first modelled by Mendelson in 1982. A model of a continuous auction was developed by Cohen et al. (1985) by only allowing limit orders at two fixed prices, buy orders at the best bid, and sell orders at the best ask. Based on this assumption they could use standard results from queuing theory to compute properties such as the average volume of stored limit orders, the expected time to execution, and the relation between the probability of execution and cancellation. Considering arbitrary order placement and cancellation processes, multiple price levels could be introduced by Domowitz and Wang (1994). These processes are time-stationary and do not respond to changes in the best bid or ask. Properties such as the distribution of the bid-ask spread, the transaction prices and waiting times for execution can be derived. An empirical test for this model was performed by Bollerslev et al. (1997) who used data for the Deutschmark/US Dollar exchange rate. It turned out that the model does a good job of predicting the distribution of the spread but does not make a prediction about price diffusion from which errors in the predictions of the spread and stored supply and demand arise.

If we now go over the models in the physics literature, we observe that dynamic issues are also addressed. Those models appear to have been developed independently from the research in the economics literature. The feedback effect between order placement and price formation has been recognized, allowing the order placement process to change in response to changes in prices. This area of research has started with a paper by Bak et al. (1997) and was then extended by Eliezer and Kogan (1998) and by Tang (1999). Limit prices of orders are placed at a fixed distance from the mid-point and then they are randomly shuffled until transactions occur. Based on this setup, similarities with a standard reaction–diffusion model in the physics literature arise and can be used. In the model by Maslov (2000), traders do not use any particular strategies when acting in the market, but they exhibit a purely random order placement which involves no strategies. This was solved analytically in the mean field limit by Slanina (2001). However, the random order placement leads to an anomalous price diffusion with Hurst exponent $H = \frac{1}{4}$, whereas real prices tend to have $H > \frac{1}{2}$. In the Maslov model the inventory of stored limit orders either goes to zero or grows without bound if we would not assume equal probabilities for limit and market order placement. This issue can somehow be resolved by including a Poisson order cancellation as was done by Challet and Stinchcombe (2001), and independently by Daniels et al. (2003). For short times, this results in the same Hurst exponent $H = \frac{1}{4}$, but asymptotically gives $H = \frac{1}{2}$. Numerical studies on fundamentalists, technical traders and noise traders placing limit orders have been performed by Iori and Chiarella (2002).

Based on the observation that empirical observations of limit order books exist it is worthwile investigating of how to construct an analytical model and setup for the order book which allows simultaneously to incorporate empirical findings and to obtain analytical results. Therefore we will propose a mathematical framework for modelling the order book and show that both empirical results can be used as input and a variety of conclusions can be drawn.

Market makers, insiders and limit order traders in a market microstructure model

Market microstructure is the study of the process and outcomes of exchanging assets under explicit trading rules. The microstructure literature analyzes how specific trading mechanisms affect the price formation process whereas much of economics abstracts from the mechanics of trading.

Beginning with research by Garman (1976), literature has focused on understanding how market prices arise given the nature of the order flow and the market-clearing mechanism. We want to point out two distinct research paradigms that emerged. Garman determines security trading prices by focusing on the nature of order flow. The dealer's optimization problem is analyzed by Stoll and Ho. Third, the effect of multiple providers of liquidity is explicitly investigated by among others, Cohen, Maier, Schwarz and Whitcomb.

Further to that inventory based approach, there has been a variety of information-based models. These models use insights from the theory of adverse selection to demonstrate how, even in competitive markets without explicit transaction costs, bid-ask spreads would exist. The beginning of this stream of research is usually credited to Bagehot (1971). Copeland and Galai (1983) consider a very simple model. A one-period model of the market maker's pricing problem is investigated given that some fraction of traders have superior information. Glosten and Milgrom (1985) include the information content of trades in their sequential trade framework, where all market participants act competitively and are assumed to be risk-neutral.

As far as strategic trader models are concerned, we want to single out the interesting studies launched by Kyle (1985). Kyle (1985) considered a sequential auction trading model in discrete time (and its limiting continuous-time model as the distance between the trading dates approaches zero) with a risk-neutral insider, a noise trader and a competitive risk-neutral market maker. In the situation of an ex post stock value that is normally distributed and known to the insider, Kyle (1985) showed the existence of a unique equilibrium. An equilibrium in this model is attained, if the market-maker's pricing rule is "rational" given the cumulative order flow, and if the risk-neutral insider maximizes his expected terminal wealth given the market-maker's pricing rule. Back (1992, 1993) formalized Kyle's continuous auction trading model, showed the existence of an equilibrium in this continuous time setting and determined the equilibrium pricing rule in a model of more general distributions (e.g. lognormal) for the ex post stock value by solving a Hamilton-Jacobi-Bellman equation.

There is also a second approach for models with heterogeneously informed economic agents, the martingale approach. This was initiated by Duffie and Huang (1986) who provided in the spirit of Harrison and Kreps (1979) a conceptual structure for such models. By requiring that the price processes are semimartingales with respect to a filtration that contains all the information flows of the heterogeneously informed agents, Duffie and Huang (1986) study the relation between price systems, that do not admit arbitrage, and martingales. More recently, Pikovsky and Karatzas (1996) and Pikovsky (1997) then studied a continuous-time diffusion model over a Brownian filtration when the insider possesses from the beginning information about the future outcome of some random variable G, e.g. the future price of a stock. Discovering the relevance of the theory of initial enlargement of filtrations to continuous-time finance models with an insider, Pikovsky and Karatzas (1996) examined for some examples of G the insider's additional expected logarithmic utility arising from the extra information. Pikovsky (1997) considered then an equilibrium version of this model and therein proved a martingale representation theorem for initially enlarged filtrations in the case of Gaussian random variables G.

There has been extensive research on trading behaviour of market makers which are perfectly competitive. This leads to a zero-profit condition which greatly simplifies the analysis of those models. However, this contradicts empirical literature quite often, see e.g. Christie and Schultz (1994), Christie et al. (1994) or Hasbrouck and Sofianos (1993) and Sofianos (1995). Those findings are supported by both static and dynamic research which investigate the effects of strategic market makers. However, most of those models have been of static nature, and, to the best of our knowledge, there has not been any investigation as far as a dynamic model is concerned, which included both a nontrivial order book and a strategic market maker. Other papers in this area are Back (1992), Bhattacharya and Spiegel (1991), Chung and Charoenwong (1998), Dutta and Madhavan (1997), Bondarenko (2001), Chakravarty and Holden (1995), Chung et al. (1999), Dennert (1993) and Glosten (1989).

Having seen that a variety of models exist, it is an exciting research topic whether one might be able to give some kind of unifying framework within which one can derive several existing models in a straightforward way. Bondarenko and Sung (2003) considered a one-period model which also included the presence of limit order traders. We will extend their model to a multiperiod model, simultaneously providing a setup which leads to several existing models and results in a way which sometimes only requires slight modifications on the underlying assumptions.

Chapter I

Diverse markets and non-equivalent measure changes

"Mathematics is not a deductive science ; that's a cliche. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork."
- Paul R. Halmos

1 Introduction

An interesting result of Fernholz in his inspiring work about stochastic portfolio theory, see the monograph [Fer02] for a detailed account, is the possibility of arbitrage in markets where no stock is ever allowed to dominate the entire market in terms of market capitalization. The associated notion of a diverse financial market was introduced and studied by Fernholz in the papers [Fer99], [Fer01] and the monograph [Fer02]. In particular, Fernholz, Karatzas and Kardaras have shown that arbitrage opportunities relative to the market portfolio exist over any given time-horizon [FKK05].

The requirement that financial markets are diverse seems to be reasonable from a regulatory point of view (otherwise we would encounter a very different society). If the market share of an existing company exceeds certain thresholds, there will be restrictions imposed on the company which try to prevent it from increasing its market share. One can also think about regulation authorities controlling mergers and acquisitions.

The purpose of this paper is to give a somewhat generic construction which yields a multitude of diverse markets. The existence of arbitrage opportunities follows then immediately by the very nature of this construction. This allows for a very transparent explanation of this phenomenon. Our main idea is as follows: we start with a non-diverse arbitrage-free market by specifying the dynamics of the price process under some local martingale measure P^0. We then construct a diverse market by changing to another probability measure Q. Since Q is absolutely continuous but not equivalent to P^0, a simple argument based on the optional decomposition theorem then yields the existence of an arbitrage opportunity. The main technical difficulty is to ensure that the market fulfils a certain non-degeneracy condition which makes the aforementioned measure change work. We can show this for some standard models including that of Fernholz, Karatzas and Kardaras [FKK05] by using a time change technique.

Furthermore, we study the dynamics of the price processes when seen under the new measure Q. We also include a brief discussion of existing approaches to the valuation of claims in case the model is complete with respect to P^0. Finally, we show how a similar change of measure technique can also be employed in currency markets where some exchange rate mechanism has been superimposed.

Arbitrage opportunities in situations governed by an absolutely continuous but non-equivalent measure change have been studied in earlier works. Gossen-Dombrowsky [GD92] (unpublished, we are grateful to H. Föllmer for providing us with this reference) studies a complete market model in which the price process is constrained to stay inside fixed boundaries. The construction of Delbaen and Schachermayer [DS95] of arbitrage possibilities in Bessel processes is also based on a similar technique.

2 Arbitrage opportunities in diverse markets

In this main section, we define the setup and show how arbitrage opportunities arise in our market. We also give the crucial non-degeneracy condition which guarantees that our approach works.

2.1 Prelude

Here we introduce some kind of pre-model. Our main model of interest will later be obtained from this by an absolutely continuous but non-equivalent measure change. Let us first specify the (preliminary) dynamics of the price processes of n risky assets. Their dynamics are governed by a probability measure P^0 living on some filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t>0}, P^0\right)$. The filtration $(\mathcal{F}_t)$ satisfies the usual assumptions of right-continuity and completeness with $\mathcal{F}_0$ being trivial.

Definition 2.1 *The price process $X = (X_i)_{1\leqslant i\leqslant n}$ is given as stochastic exponential $\mathcal{E}(M)$ of some n-dimensional continuous local P^0-martingale M. We therefore have*

$$\frac{dX_i(t)}{X_i(t)} = dM_i(t), \qquad 1 \leqslant i \leqslant n, \qquad t \geqslant 0.$$

The market so far is directly modelled under some martingale measure P^0 for X. In particular, this excludes arbitrage opportunities. The set of all probability measures equivalent to P^0 such that X is a local martingale will be denoted by $\mathcal{M}^e(X)$. As we do not necessarily assume that the market is complete, $\mathcal{M}^e(X)$ need not be a singleton. We assume that each company has one single share outstanding. Then we can define the relative market weights:

Notation 2.2 *The relative market weight μ_i of the i-th stock is given as*

$$\mu_i(t) = X_i(t) / \left(X_1(t) + \ldots + X_n(t)\right).$$

The largest market weight is denoted by $\mu_{\max}(t) = \max_{1\leqslant i\leqslant n} \mu_i(t)$.

The following notion of diversity was introduced by Fernholz [Fer99].

Definition 2.3 *We fix a finite time horizon $T > 0$ and say that the market is* diverse *(up to time T) if there exists $\delta \in (0,1)$ such that for every $i = 1, \dots, n$*

$$\mu_i(t) < 1 - \delta, \qquad \forall\, t \in [0,T] \qquad P^0 - \text{a.s.}$$

We impose one additional condition on X (the **ND** stands for 'non-degenerate') :

Assumption ND We have for some $T > 0$ and $\delta \in (0,1)$ that

$$\begin{aligned} 0 &< \inf_{P \in \mathcal{M}^e(X)} P\left(\sup_{0 \leqslant t \leqslant T} \mu_{\max}(t) \geqslant 1 - \delta\right), \\ 1 &> P^0\left(\sup_{0 \leqslant t \leqslant T} \mu_{\max}(t) \geqslant 1 - \delta\right). \end{aligned}$$

This implies in particular that the market is *not* diverse under P^0. Later we shall give a sufficient condition for Assumption **ND** to hold and show that it is satisfied in the standard Itô model as studied in Fernholz, Karatzas and Kardaras [FKK05].

2.2 Construction of diverse markets and the arbitrage opportunity

We now pass over to a diverse market, governed by a probability measure Q which we shall construct using a certain change of measure technique. Under this measure Q, we will be able to show arbitrage opportunities. For this we first have to define what we mean by arbitrage. Here we use the notion of arbitrage with respect to (general) admissible strategies as defined in Delbaen and Schachermayer [DS95].

Definition 2.4 (Arbitrage opportunity) *A predictable process H that is X-integrable for a semimartingale X is called admissible if $\int H\,dX$ is uniformly bounded from below. The semimartingale X satisfies the no-arbitrage property for admissible integrands under Q if H admissible and $\int_0^T H_t\,dX_t \geqslant 0$ Q-a.s. imply $\int_0^T H_t\,dX_t = 0$ Q-a.s.*

We can now define our measure change which directly leads to the construction of diverse markets:

Definition 2.5 *Assume* **ND**. *We define a probability measure Q absolutely continuous to P^0 via its Radon-Nikodym density*

$$\frac{dQ}{dP^0} = \begin{cases} 0 & \text{if } \mu_{\max}(t) \geqslant 1 - \delta \text{ for some } t \in [0,T] \\ c & \text{else} \end{cases} \tag{2.1}$$

where c is a normalizing constant. Since

$$P^0\left(\sup_{0 \leqslant t \leqslant T} \mu_{\max}(t) \geqslant 1 - \delta\right) < 1$$

by **ND**, *we can always find $c \in \mathbb{R}$ such that Q is a probability measure because dQ/dP^0 is not P^0-a.s. equal to 0. As*

$$P^0\left(\sup_{0 \leqslant t \leqslant T} \mu_{\max}(t) \geqslant 1 - \delta\right) > 0$$

by **ND**, *Q is absolutely continuous with respect to P^0 but not equivalent. This is crucial for the existence of arbitrage opportunities.*

Remark 2.6 *The filtration $(\mathcal{F}_t)$ typically does not satisfy the usual conditions with respect to Q. However, we refer to the Remark following Theorem 1 in Delbaen and Schachermayer [DS95] for a remedy: consider the filtration $(\mathcal{G}_t)$ obtained from $(\mathcal{F}_t)$ by adding all Q-null sets. The results in [DS95] then show that whenever we have a stopping time τ_Q and a $(\mathcal{G}_t)$-predictable process H_Q there exists a stopping time τ_P and a $(\mathcal{F}_t)$-predictable process H_P such that Q-a.s. $\tau_Q = \tau_P$ and H_Q and H_P are Q-indistinguishable. This implies that whenever we need to work with a $(\mathcal{G}_t)$-predictable process we can essentially replace it by an $(\mathcal{F}_t)$-predictable process. We shall always do so without further notice.*

The dynamics of X with respect to Q can be described via Lenglart's extension of Girsanov's theorem, which we will discuss in Section 3. Hence we get many examples of diverse markets by our construction: every price process X as above, satisfying **ND**, leads to a diverse market when seen under Q.

Let us now construct an arbitrage opportunity under Q via an admissible strategy by using an extension of the argument given in Gossen-Dombrowsky [GD92] and Delbaen and Schachermayer [DS95] and applying it to our setting of incomplete markets. We shall make use of the following optional decomposition theorem, see Föllmer and Kramkov [FK97].

Theorem 2.7 ***(Optional decomposition theorem)*** *Consider a process V which is bounded from below and a P-supermartingale for all $P \in \mathcal{M}^e(X)$ (which is non-empty since $P^0 \in \mathcal{M}^e(X)$). Then there exists a predictable X-integrable process H and an increasing adapted process C with $C_0 = 0$ such that*

$$V = V_0 + \int H\,dX - C.$$

We now state the main result about arbitrage opportunities:

Proposition 2.8 *Consider a probability measure Q which is absolutely continuous but not equivalent to P^0. If*

$$\sup_{P\in\mathcal{M}^e(X)} P\left(\frac{dQ}{dP^0} > 0\right) < 1 \tag{2.2}$$

then there exists an arbitrage opportunity under Q which can be realized via an admissible strategy.

Remark 2.9 *Note that it follows from Assumption* **ND** *that Q as in Definition 2.5 is an absolutely continuous, non-equivalent probability measure (with respect to P^0) and fulfils Condition (2.2).*

Proof. We consider the claim

$$f := \mathbf{1}_{\left\{\frac{dQ}{dP^0}>0\right\}}$$

and define the process $V = (V_t)_{t\geqslant 0}$ as

$$V_t = \operatorname{ess}\sup_{P\in\mathcal{M}^e(X)} E_P\left(f \mid \mathcal{F}_t\right), \qquad t \geqslant 0.$$

V is a P-supermartingale for all $P \in \mathcal{M}^e(X)$, see Föllmer and Kramkov [FK97]. It follows that for $t \geqslant 0$,

$$V_t \geqslant E_{P^0}(f \mid \mathcal{F}_t) \geqslant 0 \qquad P^0 - \text{a.s.}$$

By the optional decomposition theorem, there exist H and C as specified above such that

$$V = V_0 + \int H\,dX - C.$$

Moreover, H is admissible since $\int H\,dX$ is bounded from below: for $t \geqslant 0$,

$$\begin{aligned}\int_0^t H_s\,dX_s &= V_t - V_0 + C_t \geqslant -V_0 \\ &= -\sup_{P \in \mathcal{M}^e(X)} E_P(f) \geqslant -1 \qquad P^0 \text{ and } Q - \text{a.s.}\end{aligned}$$

For H to be an arbitrage opportunity under Q, we need to check whether

$$\int_0^T H_s\,dX_s = f - V_0 + C_T > 0 \qquad Q - \text{a.s.}$$

Since we have that $f = 1$ Q-a.s., $C_0 = 0$ and C is increasing, this holds in particular if $1 - V_0 > 0$ or

$$\sup_{P \in \mathcal{M}^e(X)} E_P(f) < 1,$$

which is equivalent to

$$\sup_{P \in \mathcal{M}^e(X)} P\left(\frac{dQ}{dP^0} > 0\right) < 1,$$

which is our Assumption (2.2). ∎

Remark 2.10 *Here we have constructed an arbitrage opportunity with respect to admissible integrands. This means that the value of the arbitrage portfolio is bounded from below in absolute terms. In relative arbitrage, as in [Fer02] or [FKK05], the arbitrage portfolio is bounded from below relative to the market portfolio, or perhaps some other well-defined portfolio. This amounts to a change in numeraire for the lower bound from (constant) riskless asset to market portfolio. In general, relative arbitrage is not the type of arbitrage we have here, since the numeraire portfolio is not necessarily bounded, so the result here does not follow from [Fer02] or [FKK05]. Moreover, although the sum of our arbitrage portfolio and the market portfolio dominates the market portfolio, it is not necessarily bounded from below relative to the market portfolio as numeraire.*

2.3 On the non-degeneracy condition

We now give a condition which guarantees **ND** for small enough time horizons T. This condition furthermore implies Condition (2.2). We recall that M is assumed to be a continuous local P^0-martingale.

Theorem 2.11 *Assume $\mu_{\max}(0) < 1-\delta$ and that there exists $0 < \varepsilon < \kappa$ such that for all $\mathbb{R}^n$-valued processes η we have*

$$\varepsilon \int_0^t \|\eta(s)\|^2 \, ds \leqslant \frac{1}{2}\sum_{j=1}^n \sum_{k=1}^n \int_0^t \eta_j(s) \, d[M_j, M_k]_s \, \eta_k(s) \leqslant \kappa \int_0^t \|\eta(s)\|^2 \, ds, \tag{2.3}$$

where $\|\cdot\|$ denotes the Euclidean norm. Then **ND** *is satisfied for some $T > 0$ small enough where the probability space might possibly have been extended to support an independent Brownian motion.*

Proof. Let $T > 0$ to be chosen later. Fix $t \in [0, T]$. We have

$$\begin{aligned} d\left(\sum_{j=1}^n X_j(t)\right) &= \left(\sum_{j=1}^n X_j(t)\right) \sum_{k=1}^n \frac{dX_k(t)}{\sum_{j=1}^n X_j(t)} \\ &= \left(\sum_{j=1}^n X_j(t)\right) \sum_{k=1}^n \mu_k(t) \, dM_k(t), \end{aligned}$$

hence

$$\sum_{j=1}^n X_j(t) = \mathcal{E}\left(\sum_{k=1}^n \int_0^t \mu_k(s) \, dM_k(s)\right).$$

Now fix some $i \in \{1, ..., n\}$ and set

$$\widetilde{M}_i(t) := M_i(t) - \sum_{j=1}^n \int_0^t \mu_j(s) \, dM_j(s) = -\sum_{j=1}^n \int_0^t \widetilde{\mu}_j(s) \, dM_j(s),$$

where

$$\widetilde{\mu}_j(t) = \begin{cases} 1 - \mu_i(t) & j = i \\ \mu_j(t) & j \neq i \end{cases}.$$

We get from our Assumption (2.3), together with $\sqrt{n}\,\|\mu\| \geqslant |\mu_1 + \cdots + \mu_n| = 1$, that

$$\begin{aligned} -\kappa t &\leqslant -\frac{1}{2}\sum_{j=1}^n \sum_{k=1}^n \int_0^t \mu_j(s) \, d[M_j, M_k]_s \, \mu_k(s) \leqslant -\frac{\varepsilon}{n} t, & (2.4) \\ \varepsilon t &\leqslant \frac{1}{2}[M_i]_t \leqslant \kappa t, & (2.5) \end{aligned}$$

and estimate

$$\begin{aligned} \mu_i(t) &< 1-\delta \\ &\iff \log X_i(t) < \log(1-\delta) + \log \sum_{j=1}^n X_j(t) \\ &\iff \widetilde{M}_i(t) < a + \frac{1}{2}[M_i]_t - \frac{1}{2}\sum_{j=1}^n \sum_{k=1}^n \int_0^t \mu_j(s) \, d[M_j, M_k]_s \, \mu_k(s), \end{aligned}$$

where $a = \log \sum_{j=1}^{n} X_j(0) - \log X_i(0) + \log(1-\delta) > 0$ since $\mu_i(0) \leqslant \mu_{\max}(0) < 1-\delta$. Therefore, with $b = \kappa - \varepsilon > 0$ and using both the left-hand sides of (2.4) and (2.5), we arrive at

$$\mu_i(t) < 1-\delta \qquad \text{if} \qquad \widetilde{M}_i(t) < a - bt. \tag{2.6}$$

$\widetilde{M}_i$ is a continuous local martingale with $\widetilde{M}_i(0) = 0$. We can estimate its quadratic variation by Assumption (2.3) as

$$\begin{aligned} \left[\widetilde{M}_i\right]_t &= \sum_{j=1}^{n}\sum_{k=1}^{n} \int_0^t \widetilde{\mu}_j(s)\, d[M_j, M_k]_s\, \widetilde{\mu}_k(s) \\ &\geqslant 2\varepsilon \int_0^t \|\widetilde{\mu}(s)\|^2\, ds \\ &\geqslant 2\varepsilon \int_0^t (1-\mu_i(s))^2\, ds \end{aligned} \tag{2.7}$$

as well as, using $\|\widetilde{\mu}\|^2 \leqslant |\widetilde{\mu}_1 + \ldots + \widetilde{\mu}_n|^2 = 4(1-\mu_i)^2$,

$$\begin{aligned} \left[\widetilde{M}_i\right]_t &\leqslant 2\kappa \int_0^t \|\widetilde{\mu}(s)\|^2\, ds \\ &\leqslant 8\kappa \int_0^t (1-\mu_i(s))^2\, ds \\ &\leqslant 8\kappa t. \end{aligned} \tag{2.8}$$

To show that $P^0\left(\sup_{0\leqslant t\leqslant T} \mu_{\max}(t) \geqslant 1-\delta\right) < 1$ we use that $\widetilde{M}_i$ is a time-changed Brownian motion. Indeed, by Karatzas and Shreve [KS91], Theorem 3.4.6 and Problem 3.4.7, there exists on a suitably extended probability space a Brownian motion B with $B_0 = 0$ such that $\widetilde{M}_i(t) = B_{[\widetilde{M}_i]_t}$ for $t \geqslant 0$. In particular, by the construction of this extension as carried out in Karatzas and Shreve [KS91], Remark 3.4.1, we can take B to be a P-Brownian motion simultaneously for all $P \in \mathcal{M}^e(X)$ (we need this to prove the other inequality in **ND**). Since $a > 0$ we can choose $T_i > 0$ small enough such that $a - bT_i \geqslant \rho$ for some $\rho > 0$ and such that

$$P^0\left(B_t < \rho \qquad \text{for all } t \in [0, 8\kappa T_i]\right) > 1 - \frac{1}{n}.$$

By using (2.8) we estimate

$$\begin{aligned} & P^0\left(\widetilde{M}_i(t) < a - bt \qquad \text{for all } t \in [0, T_i]\right) \\ =\; & P^0\left(B_{[\widetilde{M}_i]_t} < a - bt \qquad \text{for all } t \in [0, T_i]\right) \\ \geqslant\; & P^0\left(B_{[\widetilde{M}_i]_t} < \rho \qquad \text{for all } t \in [0, T_i]\right) \\ =\; & P^0\left(B_t < \rho \qquad \text{for all } t \in \left[0, \left[\widetilde{M}_i\right]_{T_i}\right]\right) \\ \geqslant\; & P^0\left(B_t < \rho \qquad \text{for all } t \in [0, 8\kappa T_i]\right) > 1 - \frac{1}{n}. \end{aligned}$$

From this and (2.6) we can conclude by setting $T = \min_{1\leqslant i\leqslant n} T_i$ that for all $i \in \{1, \ldots, n\}$

$$P^0\left(\sup_{0\leqslant t\leqslant T} \mu_i(t) < 1-\delta\right) > 1 - \frac{1}{n},$$

hence

$$P^0\left(\sup_{0\leqslant t\leqslant T}\ \mu_{\max}(t) < 1-\delta\right) > 0,$$

which is equivalent to

$$P^0\left(\sup_{0\leqslant t\leqslant T}\ \mu_{\max}(t) \geqslant 1-\delta\right) < 1,$$

as desired.

To show that $\inf_{P\in\mathcal{M}^e(X)} P\left(\sup_{0\leqslant t\leqslant T}\ \mu_i(t)\geqslant 1-\delta\right) > 0$ we use the lower bound (2.7) for $\left[\widetilde{M}_i\right]$. Moreover, here the time horizon $T>0$ can be arbitrary. Similarly as before, we use the right-hand sides of (2.4) and (2.5) to show (now with $b=\kappa-\varepsilon/n>0$) that $\widetilde{M}_i(t)\geqslant a+bt$ implies $\mu_i(t)\geqslant 1-\delta$. We consider the following two cases.

Case 1

$$\inf_{P\in\mathcal{M}^e(X)} P\left(\int_0^T (1-\mu_i(s))^2\, ds \leqslant \delta^2 T\right) > 0$$

for some $T>0$. As

$$\begin{aligned}(1-\mu_i(t))^2 &\leqslant \delta^2 && \text{for some } t\in[0,T]\\ \Longleftrightarrow\ \mu_i(t) &\geqslant 1-\delta && \text{for some } t\in[0,T]\end{aligned}$$

it follows that

$$\inf_{P\in\mathcal{M}^e(X)} P\left(\mu_i(t)\geqslant 1-\delta \text{ for some } t\in[0,T]\right) > 0.$$

This is what we wanted to show.

Case 2

$$\inf_{P\in\mathcal{M}^e(X)} P\left(\int_0^T (1-\mu_i(s))^2\, ds \leqslant \delta^2 T\right) = 0$$

for all $T>0$. Consider a minimizing sequence $(P_n)\subset\mathcal{M}^e(X)$ such that

$$\begin{aligned}&P_n\left(\widetilde{M}_i(t)\geqslant a+bt \text{ for some } t\in[0,T]\right)\\ \searrow\ &\inf_{P\in\mathcal{M}^e(X)} P\left(\widetilde{M}_i(t)\geqslant a+bt \text{ for some } t\in[0,T]\right).\end{aligned}$$

We may assume that

$$\lim_{n\to\infty} P_n\left(\int_0^T (1-\mu_i(s))^2\, ds \leqslant \delta^2 T\right) = 0 \tag{2.9}$$

(otherwise we are either in Case 1 (for (P_n)) or can extract a further subsequence fulfilling (2.9)). Again we proceed by time-changing the process $\widetilde{M}_i$ into a Brownian motion B as above:

$$\begin{aligned}
& \inf_{P \in \mathcal{M}^e(X)} P\left(\widetilde{M}_i(t) \geqslant a + bt, \text{ for some } t \in [0,T]\right) \\
= & \lim_{n\to\infty} P_n \left(B_{[\widetilde{M}_i]_t} \geqslant a + bt, \text{ for some } t \in [0,T]\right) \\
\geqslant & \lim_{n\to\infty} P_n \left(B_{[\widetilde{M}_i]_t} \geqslant a + bT, \text{ for some } t \in [0,T]\right) \\
= & \lim_{n\to\infty} P_n \left(B_t \geqslant a + bT, \text{ for some } t \in \left[0, \left[\widetilde{M}_i\right]_T\right]\right) \\
\geqslant & \lim_{n\to\infty} P_n \left(B_t \geqslant a + bT, \text{ for some } t \in \left[0, 2\varepsilon \int_0^T (1-\mu_i(s))^2 \, ds\right]\right) \\
\geqslant & \lim_{n\to\infty} P_n \left(B_t \geqslant a + bT, \text{ for some } t \in \left[0, 2\varepsilon\delta^2 T\right], \int_0^T (1-\mu_i(s))^2 \, ds > \delta^2 T\right) \\
> & \; 0,
\end{aligned}$$

where the last inequalities follow by (2.7), (2.9) and the fact that

$$P_n \left(B_t \geqslant a + bT \text{ for some } t \in \left[0, 2\varepsilon\delta^2 T\right]\right)$$

does not depend on n. This gives us our second result.

■

Remark 2.12 *We cannot guarantee the existence of an arbitrage opportunity if the number of stocks in the portfolio goes to infinity. In that case we cannot choose by our construction a nonzero T such that* **ND** *holds up to T. This justifies a criticism of D. Hobson which is based on the fact that one can often observe that new firms are created once one company enjoys a dominant position in some market.*

2.4 On the non-degeneracy condition in the standard Itô model

Here we will show that Condition 2.3 is satisfied in the standard Itô model as used in Fernholz, Karatzas and Kardaras [FKK05]. This shows in particular that their assumptions imply Assumption **ND** and hence the existence of arbitrage opportunities after the measure change to Q. The prices of n stocks are modelled by the following linear stochastic differential equation:

$$\frac{dX_i(t)}{X_i(t)} = b_i(t)\,dt + \sum_{\nu=1}^{m} \xi_{i\nu}(t)\, dW_\nu(t), \qquad X_i(0) = x_i, \qquad t \in [0,\infty),$$

for $i = 1, \ldots, n$, where W is a standard m-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$, $m \geq n$. The coefficients are assumed to be adapted and finite; moreover,

$$\int_0^t \|b(s)\|^2 \, ds < \infty, \quad \forall t \in (0,\infty).$$

Denote $\sigma = \xi\xi'$ where $\xi = (\xi_{i\nu})_{1\leqslant i\leqslant n, 1\leqslant \nu\leqslant m}$. We shall need the crucial (but common) assumption that the market is *non-degenerate* (in the terminology of Fernholz [Fer99]), i.e.

$$x'\sigma(t)\,x \geqslant \varepsilon\,\|x\|^2, \qquad \forall x \in \mathbb{R}^n, \qquad \forall t \in [0,\infty), \tag{2.10}$$

and has *bounded variance*

$$x'\sigma(t)\,x \leqslant M\,\|x\|^2, \qquad \forall x \in \mathbb{R}^n, \qquad \forall t \in [0,\infty) \tag{2.11}$$

for some real constants $M > \varepsilon > 0$.

Condition (2.10) allows us to remove the drift terms b_i using Girsanov's theorem so that we get the following dynamics of the price processes:

$$\frac{dX_i(t)}{X_i(t)} = \sum_{\nu=1}^{m} \xi_{i\nu}(t)\, dB_\nu(t), \qquad i = 1,\ldots,n, \qquad t \in [0,\infty),$$

where B is a standard m-dimensional Brownian motion under some martingale measure P^0 (see Section 5.8 of Karatzas and Shreve [KS91]).

To apply the previous results, we need to show Condition 2.3. In our case this amounts to the existence of two numbers $0 < \varepsilon < \kappa$ such that for all $\mathbb{R}^n$-valued processes η we have

$$\varepsilon \int_0^t \|\eta(s)\|^2\, ds \leqslant \frac{1}{2} \sum_{j=1}^{n}\sum_{k=1}^{n} \int_0^t \eta_j(s) \sum_{\nu=1}^{m} \xi_{j\nu}(s)\,\xi_{k\nu}(s)\, ds\, \eta_k(s) \leqslant \kappa \int_0^t \|\eta(s)\|^2\, ds.$$

The left inequality follows from the non-degeneracy condition (2.10) while the right-hand side follows because of the condition of bounded variance (2.11).

Remark 2.13 *It is apparent from the preceding discussion that the validity of Assumption* **ND**, *and hence the existence of the arbitrage opportunity, depends crucially on the conditions (2.10), (2.11) of non-degeneracy and bounded variance. While it seems to be reasonable from an economic point of view to assume that actual markets are diverse, and that the regulatory impact ensuring diversity corresponds in the mathematical model to the measure change to Q, a stalwart of the efficient market hypothesis might object to (pre-)models where those conditions are fulfilled.*

3 The dynamics of the price processes under Q. Further applications.

This section has been included for the convenience of the reader: we review related results and put them into the context of our setting.

3.1 Q-Dynamics of the price processes

Fernholz, Karatzas and Kardaras [FKK05] construct an explicit example of price processes which lead to a diverse market. With our approach we can generate a multitude of diverse markets: every pre-model as in section 2.1 satisfying **ND** leads to a diverse market when seen under Q as defined in (2.1). Let us now illustrate the new dynamics under the measure Q. For this, we shall make use of Lenglart's extension of Girsanov's theorem, see Lenglart [Len77].

Theorem 3.1 ***(Lenglart's theorem)*** *Let Q be a probability measure absolutely continuous with respect to P^0. Define the process Z as*

$$Z_t = E_{P^0}\left(\frac{\chi_{\{\frac{dQ}{dP^0}>0\}}}{P^0\left(\frac{dQ}{dP^0} > 0\right)} \middle| \mathcal{F}_t \right).$$

Let X be a continuous local martingale under P^0. Then there exists an X-integrable process α such that

$$X - \int \frac{1}{Z_-} d[Z,X] = X - \int \alpha \, d[X]$$

is a Q-local martingale.

Although it seems in general difficult to find an explicit expression for the drift α in our situation, let us recall from Jacod [Jac79] a more detailed description via the Kunita-Watanabe decomposition. In our case, the price processes are given as

$$\frac{dX(t)}{X(t)} = dM(t), \qquad t \geqslant 0,$$

where M is a continuous local martingale under P^0. We note that the process Z in Lenglart's theorem is square-integrable. Using the Galtchouk-Kunita-Watanabe decomposition, we project Z on the space of all square-integrable martingales which can be written as $\int \gamma \, dM$ for some predictable process γ and write the resulting orthogonal projection as $\int \beta \, dM$. The process $\int \beta \, dM$ is square-integrable by construction and hence we have a fortiori that

$$\int_0^T \beta'(t) \, d[M]_t \, \beta(t) < \infty \qquad P^0 - \text{a.s.}$$

Moreover, $[Z, M] = \int \beta \, d[M]$. Z is Q-a.s. strictly positive (see Revuz and Yor [RY04], Proposition VIII.1.2.), so we may set

$$\alpha = \frac{\beta}{Z_-}.$$

It results that

$$\begin{aligned} \int \frac{1}{Z_-} d[Z,M] &= \int \frac{\beta}{Z_-} d[M] \\ &= \int \alpha \, d[M] \end{aligned}$$

and

$$\int_0^T \alpha'(t) \, d[M]_t \, \alpha(t) = \int_0^T \beta'(t) \frac{1}{Z_{t-}^2} d[M]_t \, \beta(t) < \infty \qquad Q - \text{a.s.} \tag{3.1}$$

Summing up, the dynamics of X under Q are given as

$$\frac{dX}{X} = d\widetilde{M} + \alpha \, d\left[\widetilde{M}\right], \tag{3.2}$$

where $\widetilde{M} = M - \int \alpha \, d[M]$ is a local Q-martingale.

3.2 Valuation of claims when the pre-model is complete

Let us now briefly discuss the problem of valuation of claims in our setting. First we observe that defining a price based on superreplication, using admissible integrands, would lead to a non-finite price in our case. Fortunately, it turns out that pricing is still possible if we only allow strategies which require no intermediate credit. We assume that the market under P^0 is complete and apply the traditional replication approach to find a price for contingent claims in our diverse market. Here we review two approaches taken in the literature and show that they are in fact equivalent.

Gossen-Dombrowsky [GD92] proposes to consider for any integrable claim H a modified claim $\widetilde{H} = H\chi_{\Omega_Q}$, where Ω_Q is the support of the measure Q. He then assigns to it the usual no-arbitrage price $E_{P^0}\left[\widetilde{H}\right]$ since P^0 is the unique martingale measure for X under the completeness assumption. This method coincides with the approach taken in Fernholz, Karatzas and Kardaras [FKK05] which, generalized to our setting, is as follows: motivated by (3.2), we consider the stochastic exponential

$$L = \mathcal{E}\left(-\int \alpha \, d\widetilde{M}\right).$$

It follows from (3.1) that L is Q-a.s. strictly positive. The proposed price for the claim H is then $E_Q\left[HL_T\right]$ which, as

$$L_T \frac{dQ}{dP^0} = \chi_{\Omega_Q} \qquad Q-\text{a.s.},$$

coincides with the price as in Gossen-Dombrowsky [GD92]. Note that for $H \geqslant 0 \quad Q$–a.s, we get for the associated value process

$$V_. = E_{P^0}\left[H\chi_{\Omega_Q}\middle|\mathcal{F}_.\right] \geqslant 0 \qquad Q-\text{a.s.}$$

3.3 Further arbitrage opportunities in stock, bond and currency markets

The original motivation of Gossen-Dombrowsky [GD92] comes from a model where the stock price follows a geometric Brownian motion respecting two a priori fixed exponential curves as upper and lower boundaries. Delbaen and Schachermayer [DS95] show that there are arbitrage possibilities in Bessel processes. Using a Brownian motion B starting in one they construct a new measure which assigns probability zero to the set of paths where B ever hits zero.

In a similar spirit, one gets arbitrage opportunities in bond and currency markets once certain bounds have been imposed. While mathematically these situations are much more straightforward to treat than the case of diverse markets, we shall still give a brief illustration of an exchange mechanism where the domestic currency is tied to some foreign currency by allowing it to float freely only within a certain range. As usual, we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P^0)$. The exchange rate process X, which is used to convert foreign payoffs into domestic currency, is modelled under P^0 for simplicity by the following stochastic differential equation:

$$\frac{dX(t)}{X(t)} = \sigma \, dW(t),$$

where σ is some positive constant and W is a P^0-Brownian motion. We now assume that by regulation, X is restricted to move only in a range of $[a,b]$ for some $b > a > 0$ (whether the regulating authority is able to support the currency in this manner is of no concern to us here, albeit it is of highly practical relevance).

Fix a finite time horizon $T > 0$. Observe that in our setup (where it is assumed that $X(0) \in (a,b)$) we have

$$P^0 \left(a \leqslant X(t) \leqslant b \qquad \forall t \in [0,T]\right) > 0$$

and

$$P^0 \left(\exists t \in [0,T] \text{ s.th. } X(t) \notin [a,b]\right) > 0.$$

We now pass over to a new measure Q (reflecting the regulatory impact) which is defined via its density

$$Z = \left.\frac{dQ}{dP^0}\right| \mathcal{F}_T = \begin{cases} 0 & \text{if } X(t) \notin [a,b] \text{ for some } t \in [0,T] \\ c & \text{otherwise} \end{cases},$$

where c is a normalizing constant. We consider the contingent claim

$$f = \chi_{\{Z>0\}}.$$

As the original market under P^0 is complete, there exists a replicating strategy which, under Q, represents an arbitrage opportunity since $Q(Z=0) = 0$. This can be seen by the same argument as in the proof of Proposition 2.8 (here it is even easier since we are in a complete-market situation).

4 Summary and outlook

Here we looked at the notion of diverse markets and introduced it into the standard models of financial markets. Using a certain change of measure technique we show how this new condition leads to arbitrage opportunities. This technique is quite powerful and can also be applied in similar situations in bond and currency markets. Further research areas arise naturally. One goal is of course to make our mathematical models of financial markets more and more realistic such that they reflect real worlds. Therefore one task is to first identify new restrictions or conditions which exist in the financial markets and then incorporate it in the stock price models if that feature is not yet reflected.

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Chapter II

The limit order book and its applications

"The bottom line for mathematicians is that the architecture has to be right. In all the mathematics that I did, the essential point was to find the right architecture. It's like building a bridge. Once the main lines of the structure are right, then the details miraculously fit. The problem is the overall design."

"Freeman Dyson: Mathematician, Physicist, and Writer". Interview with Donald J. Albers, The College Mathematics Journal, vol. 25, no. 1, January 1994.

1 Introduction

In recent years we can observe that more and more of the major exchanges in the world rely upon limit orders for the provision of liquidity. Data has only been available for the last few years and some empirical investigations have been undertaken in the literature, see among many others [BMP02] and [PB03a]. Of course, knowing the entire order book gives much more information than just knowing the history of the stock price process. Therefore, one will want to be able to exploit this additional information and take it into account when looking at the stock market.

Electronic trading with the help of a public limit order book continues to increase its share in worldwide security trading. In order-driven markets investors can submit either market or limit orders. Market orders are typically submitted by impatient traders, since those are immediately executed against the quoted bid or ask. On the other hand, patient traders may prefer to submit limit orders to guarantee that orders are executed only when the market price is below or above a certain threshold. Limit orders are stored in the book of the exchange and executed according to different rules which vary from exchange to exchange. The most important aspects are time priority, given a certain price and price priority across different price levels. The upward price movements of a rising market trigger limit orders to sell; if the market is falling, the downward movements trigger limit orders to buy. This also implies that limit orders provide liquidity and immediacy to market orders. The advantage of using limit orders is that, by delaying transactions, patient traders may be able to trade at a more favourable price. However we have to deal with the uncertainty as to when the orders are executed and if they are executed at all. In addition, the order may be cancelled.

This can be caused by the trader's perception about the fair price of an asset, which may have changed since the time the order was placed.

An excellent overview of the literature on order books is given in [SFGK03]. Here we summarize those findings. In 1985, Bias et al. [BHS95] were one of the first that looked into the real-world limit order markets. They conduct a comprehensive empirical study of the order flow and the limit order book at the Paris Bourse and develop a theoretical model of limit order submissions. Different types of traders exhibit different behaviour in the limit order markets. Among others, Lo et. al.[LMZ02] and Hollifield et. al. [HMS04] try to explain the rationale behind this. Many authors have done research on the microstructure of double auction markets, limit order books, order flow, traded volume and the bid-ask spread. Among many other papers, one can find results in [BDW97], [BMP02], [CI02], [CS01], [DFIS03], [DW94], [GPGS00], [JGL94], [JKL94], [Mas00], [MM01], [PB03b] and [Sla01].

There are two independent lines of prior work, one in the financial economics literature and the other in the physics literature. In the economics literature, we mainly deal with a static order process and base the models base on econometrics. On the other hand, the models in the physics literature are mostly artificial toy models, but they are fully dynamic since they allow the order process to react to changes in prices. Random order placement with periodic clearing was first modelled by Mendelson [Men82]. A model of a continuous auction was developed by Cohen et al. [CCM85] by only allowing limit orders at two fixed prices, buy orders at the best bid, and sell orders at the best ask. Based on this assumption they could use standard results from queuing theory to compute properties such as the average volume of stored limit orders, the expected time to execution, and the relation between the probability of execution and cancellation. Considering arbitrary order placement and cancellation processes, multiple price levels could be introduced by Domowitz and Wang [DW94]. These processes are time-stationary and do not respond to changes in the best bid or ask. Properties such as the distribution of the bid-ask spread, the transaction prices and waiting times for execution can be derived. An empirical test for this model was performed by Bollerslev et al. [BDW97] who used data for the Deutschmark/US Dollar exchange rate. It turned out that the model does a good job of predicting the distribution of the spread but does not make a prediction about price diffusion from which errors in the predictions of the spread and stored supply and demand arise.

If we now go over the models in the physics literature, we observe that dynamic issues are also addressed. Those models appear to have been developed independently from the research in the economics literature. The feedback effect between order placement and price formation has been recognized, allowing the order placement process to change in response to changes in prices. This area of research has started with a paper by Bak et al. [BBS97] and was then extended by Eliezer and Kogan [EI98] and by Tang [TT99]. Limit prices of orders are placed at a fixed distance from the mid-point and then they are randomly shuffled until transactions occur. Based on this setup, similarities with a standard reaction-diffusion model in the physics literature arise and can be used. In the model by Maslov [Mas00], traders do not use any particular strategies when acting in the market, but they exhibit a purely random order placement which involves no strategies. This was solved analytically in the mean field limit by Slanina [Sla01]. However, the random order placement leads to an anomalous price diffusion with Hurst exponent $H = \frac{1}{4}$, whereas real prices tend to have $H > \frac{1}{2}$. In the Maslov model the inventory of stored limit orders either goes to zero or grows without bound if we would not assume equal probabilities for limit and market order placement. This issue can somehow be resolved by including a Poisson order cancellation as was done by Challet and Stinchcombe [CS01], and independently by Daniels et al. [DFIS03]. For short times, this results in the same Hurst exponent $H = \frac{1}{4}$, but asymptotically gives $H = \frac{1}{2}$. Numerical studies on fundamentalists, technical traders and noise traders placing limit orders have been performed by Iori and Chiarella [CI02].

We propose the following setup to model the limit order book. The first goal is to find a suitable framework for the order arrival process and the order book. We start with the description of an exogenously given reference price which we call transaction price. It evolves according to a semimartingale and in a few cases will be further specialized to an Itô process or even a geometric Brownian motion. Limit orders are submitted relative to this price according to the following rule: Buy orders are only allowed to be placed below the transaction price and sell orders above this price. Next, we model the arrival process of limit orders. For this, we use the concept of random measures. A random measure here describes the number of orders which arrive in a certain subset of a three-dimensional space which is spanned by time, relative limit order price and limit order size. Of course, the number of orders is random. We are free to use empirical observations to describe the arrival process of orders, in particular the time between orders, their price relative to the current transaction price and their size. From this we can go over to modelling the order book itself. This means we need to include the absolute price of the orders, which can again be solved by using a random measure (now in a four-dimensional space). Clearly, we have to take the time evolution into account. No sell orders can be below the transaction price at any point in time and no buy orders can be above this price. We describe this by assuming that limit orders are executed once the reference price reaches them.

It will turn out that the order book can be described as the difference of two doubly stochastic Poisson processes at every point in time. Another characterization can be given as the difference of two infinite sums of Bernoulli distributed random variables. Now we have implemented the framework for the limit order book.

Three fundamental assumptions were made: Buy orders are only placed below the transaction price, sell orders above, limit order submission does not influence the reference price, and orders are executed once they are reached by the transaction price. The last point implies in particular that we do not consider time priority or the possibility of partial execution. The last two assumptions are reasonable if we consider only small investors. Therefore we use a small-investor model which is just defined by the validity of those two assumptions. This can be compared to standard models of the stock price where we also usually consider small investors, also called price-takers in that context.

Now we start with the description of some results and observations in the small investor model.

Empirical observations lead to the result that the distribution of the distance between the limit order price and the stock price at the time of submission is heavy-tailed. From this, we derive conditions under which this property of heavy-tailedness is invariant over time, i.e. holds for the actual orders which are in the order book, not only at time of submission.

Another interesting aspect is the relation between the volatility of the stock price and the volume of orders in the order book. It actually turns out, that it is, under certain assumptions, not so much the volatility of the stock price but the behaviour of the maximum of the stock price which plays the important role here: The less fluctuation the maximum shows, the higher is the volume of limit orders. This is an aspect which would be worth investigating empirically. As a small side observation, it turns out that the intensity measure of the order book is related to a certain exotic option, a combination of a lookback option and an Asian option.

Observe that we did not start with modelling the bid and ask prices. However, once we know the order book, we can derive the distributions of the bid- and ask prices. By choosing certain parameters accordingly, it would be possible to fit the bid-ask spread to empirical data.

In the real world, cancellation plays an important role. We show how to incorporate that behaviour in our model and give natural conditions under which the above results hold in that case as well.

Furthermore, we introduce the concepts of time horizons and execution probability. Our approach allows us to calculate the time horizons of limit order traders and the corresponding distribution of the execution probability of limit orders.

Before we construct a model of the order book, we want to lay down several properties which have been empirically investigated and which would be desirable features of any theoretical model.

Therefore we collect a number of facts and empirical results which are reported in the literature. The results can be found in [BDW97], [BMP02], [CS01] and [PB03a], among many others. The most important findings are described in the sequel. The distribution of the limit price relative to the best bid should decay as a power law. If we have exponents larger than one, this results in an order book with a finite number of orders. In this case, there is a positive probability that a single market order can clear the entire market. The unconditional cumulative distribution of relative limit prices decays roughly as a power law with exponent approximately -1.6. Here we only count the number of orders and do not consider their size. The distribution of the order size should be approximately like a log-normal distribution with a power law tail. The order flow should have trends. If the order flow has been skewed towards buying, then it should be more likely that it continues to be skewed. Relative limit price levels are positively correlated with and are led by price volatility. The average order book has a maximum away from the current bid/ask and a tail reflecting the statistics of the incoming orders [BMP02]. The distribution of volume at the bid (or ask) follows a gamma distribution. The unconditional limit order size is distributed uniformly in log-size (between 10 and 50000 ticks, see [BMP02]). Let Δ be the distance between the order price and the current price. The conditional average volume is roughly independent of Δ between 1 and 20 ticks, but decays as a power law $\Delta^{-\nu}$ with ν approximately 1.5 beyond some Δ^*. The fluctuation of volume in the book as a function of Δ, defined as:

$$\sigma_V(\Delta) = \sqrt{E(V^2|\Delta) - E^2(V|\Delta)}$$

has the following property: $\sigma_V(\Delta)$ is of order 1 for $\Delta = 1$ and is roughly constant up to $\Delta = 50$.

In our model, we have several degrees of freedom in choosing certain distributions. This allows us to use empirically observed distributions as the input parameters of our model and therefore to reproduce many of the empirical findings from above.

Remark 1.1 *One of the properties of our model is that orders are filled once they are reached by the stock price. This is clearly one of our assumptions in the "small-investor" model* *which is not entirely satisfied in real markets. However, there are interesting publications by Jones et al. [JGL94], [JKL94] and by Gopikrishnan et al. [GPGS00], where the relationship between the number of trades and volatility has been investigated. In regressions of volatility on both volume and the number of transactions, the volatility-volume relation is rendered statistically insignificant. That is, it is the occurrence of transactions per se, and not their size, that generates volatility. Another approach in microstructure models, which is quite often used, is one where different kinds of traders are considered. We mention here e.g. the paper by Chiarella and Iori [CI02] where the demands of traders are assumed to consist of three components, a fundamentalist component, a chartist component and a noise-induced component. We want to stress here that our model is of a different category, since all those components are already included in either the fluctuations of the stock price or the order arrival and cancellation process.*

2 From the arrival process to the order book

Now we are ready to build a model of the order book. Before we start with the arrival process of buy and sell orders we look at some necessary preliminaries, in particular the notion of random measures. From this we can derive the state of the order book at every point in time. Here we concentrate on the sell order book, the collection of all sell orders which have not been cancelled or executed. The derivation of the buy order book is similar. Let us begin with the crucial concept of random measures.

2.1 Necessary preliminaries - random measures

One of our main tools to describe the order book will be random measures. For the convenience of the reader we recall the definition of (Poisson) random measures here, since they will be used extensively in the sequel.

Definition 2.1 (Random measure) *Given a probability space $(\Omega, \mathcal{F}, P)$ consider an arbitrary measurable space $(S, \mathcal{S})$. A random measure is defined as a $\sigma-$finite kernel μ from $(\Omega, \mathcal{F}, P)$ into S. The intensity of μ is defined as the measure $E(\mu(B))$, $B \in \mathcal{S}$.*

Remark 2.2 *We can think of μ as a random element in the space $\mathcal{M}(S)$ of $\sigma-$finite measures on S, endowed with the $\sigma-$field generated by the projection maps $\pi_B : \mu \rightarrow \mu(B)$ for arbitrary $B \in \mathcal{S}$.*

For each càdlàg process we can define a jump measure: Given a càdlàg process $(X_t)_{t\in[0,T]}$ with values in $\mathbb{R}^d$ associate with it a random measure J_X on $[0,T] \times \mathbb{R}^d$ called the jump measure in the following manner: X has at most a countable number of jumps:

$$\{t \in [0,T] : \Delta X_t = X_t - X_{t-} \neq 0\}.$$

Its elements can be arranged in a sequence $(T_n)_{n\geqslant 1}$ (not necessarily increasing) which are the (random) jump times of X. At time T_n, the process X has a discontinuity of size $Y_n = X_{T_n} - X_{T_{n-}} \in \mathbb{R}^d \setminus \{0\}$ which contains all information about the jump of the process

X: the jump times T_n and the jump sizes Y_n. The associated random measure, which we denote by J_X, is called the jump measure of the process X :

$$J_X(\omega, \cdot) = \sum_{n\geqslant 1} \delta_{(T_n(\omega),\, Y_n(\omega))} = \sum_{t\in[0,T]}^{\Delta X_t \neq 0} \delta_{(t,\, \Delta X_t)}.$$

In intuitive terms, for any measurable subset $A \subset \mathbb{R}^d : J_X([0,t] \times A) =$ number of jumps of X occuring between 0 and t whose amplitude belongs to A.

We define the special class of Poisson random measures.

Definition 2.3 (Poisson random measure) *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $E \subset \mathbb{R}^d$ and μ a given (positive) Radon measure on $(E, \mathcal{E})$. A* ***Poisson random measure*** *on E with intensity measure μ is an integer valued random measure:*

$$\begin{array}{rcl} M & : & \Omega \times \mathcal{E} \to \mathbb{N} \\ (\omega, A) & \mapsto & M(\omega, A) \end{array}$$

such that

1. *For almost all $\omega \in \Omega$, $M(\omega; \cdot)$ is an integer-valued Radon measure on E : for any bounded measurable $A \subset E$, $M(A) < \infty$ is an integer valued random variable.*

2. *For each measurable set $A \subset E$, $M(\cdot, A) = M(A)$ is a Poisson random variable with parameter $\mu(A)$:*

$$\forall k \in \mathbb{N},\ P(M(A) = k) = e^{-\mu(A)} \frac{(\mu(A))^k}{k!}.$$

3. *For disjoint measurable sets $A_1, \ldots, A_n \in \mathcal{E}$, the variables $M(A_1), \ldots, M(A_n)$ are independent.*

Theorem 2.4 *Let M be a Poisson random measure with intensity measure μ. Then the following formula holds for every measurable set B such that $\mu(B) < \infty$ and for all functions f such that $\int_B e^{f(x)} \mu(dx) < \infty$:*

$$E\left[\exp\left\{\int_B f(x)\, M(dx)\right\}\right] = \exp\left\{\int_B \left(e^{f(x)} - 1\right) \mu(dx)\right\}.$$

A Poisson random measure M on $E = [0,T] \times \mathbb{R}^d \backslash \{0\}$ can be described as the counting measure associated to a random configuration of points $(T_n, Y_n) \in E$:

$$M = \sum_{n\geqslant 1} \delta_{(T_n, Y_n)}.$$

Intuitively, each point $(T_n(\omega), Y_n(\omega)) \in E$ corresponds to an observation made at time T_n and described by a (nonzero) random variable $Y_n(\omega) \in \mathbb{R}^d$. We say that M is a nonanticipating Poisson random measure if

- $(T_n)_{n\geqslant 1}$ are nonanticipating random times.

- Y_n is revealed at T_n : Y_n is $\mathcal{F}_{T_n}$−measurable.

Definition 2.5 (Marked point process) *A **marked point process** on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is a sequence $(T_n, Y_n)_{n \geqslant 1}$ where*

- *$(T_n)_{n \geqslant 1}$ is an increasing sequence of nonanticipating random times with $T_n \to \infty$ almost surely as $n \to \infty$.*
- *$(Y_n)_{n \geqslant 1}$ is a sequence of random variables taking values in E.*
- *The value of Y_n is revealed at T_n : Y_n is $\mathcal{F}_{T_n}$−measurable.*

A Poisson random measure on E can also be considered as a random variable taking values in $\mathcal{M}(E)$, the set of Radon measures on E, on which a topology is defined as follows: a sequence μ_n of Radon measures on $E \subset \mathbb{R}^d$ is said to converge to a Radon measure μ if for any $f : E \to \mathbb{R}$ with compact support $\int f d\mu_n \to \int f d\mu$.

We close this section with a result on the convergence of Poisson random measures.

Proposition 2.6 *Let $(M_n)_{n \geqslant 1}$ be a sequence of Poisson random measures on $E \subset \mathbb{R}^d$ with intensities $(\mu_n)_{n \geqslant 1}$. Then $(M_n)_{n \geqslant 1}$ converges in distribution if and only if the intensities (μ_n) converge to a Radon measure μ. Then $M_n \Longrightarrow M$ where M is a Poisson random measure with intensity μ.*

2.2 The transaction price

Fix a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,\infty)}, P\right)$. The filtration $(\mathcal{F}_t)_{t \in [0,\infty)}$ is assumed to satisfy the usual conditions of right continuity and completeness. The transaction price process follows general semimartingale dynamics. In particular cases, we will explicitly consider the following dynamics:

$$\frac{dX_t}{X_t} = \mu_t dt + \sigma_t dW_t, \qquad X_0 = x, \qquad t \in [0, \infty), \tag{2.1}$$

where W is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$. The coefficients are such that a unique strong solution to (2.1) exists. In a few cases we also consider a geometric Brownian motion, i.e.

$$\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t, \qquad X_0 = x, \qquad t \in [0, \infty), \ \mu \in \mathbb{R}, \sigma \in \mathbb{R}_+. \tag{2.2}$$

We assume that the stock price (transaction price) X_t is exogenously given and is the reference price relative to which the limit orders are submitted. Furthermore, at time t, we do not allow limit buy orders to be higher than the current transaction price X_t and limit sell orders to be lower than X_t. We assume that those orders are immediately executed and cause part of the fluctuation of X. Our primary interest lies in the limit order book. Therefore, market orders are not explicitly considered. However, their cumulative effect is reflected in the dynamics of the stock price process.

2.3 The arrival process of limit orders and the order book

After the description of the reference price, we now go over to our main task, describing the arrival process of orders and the order book itself.

2.3.1 The general setup

We want to calculate the distribution of the sell order book under suitable assumptions on the process $(X_t)_{t\geqslant 0}$.

The distribution of the order book can be derived if we make reasonable assumptions on the order arrival times, relative limit prices and order sizes. The choice of reasonable distributions should be based on empirical data, as described before. We start with introducing the relevant objects.

Notation 2.7 (Arrival order book) *We use the following notation for the* ***arrival sell order book****, conditional on a realization of the stock price process up to time* T *(denoted by* θ*),*

$$\mathcal{A}^s_\theta : \Omega \times \mathcal{B}\left([0,T] \times \mathbb{R}^2_+\right) \to \mathbb{R}$$

where

$$\mathcal{A}^s_\theta\left([0,t] \times [0,p_r] \times [0,s]\right), \qquad t, p_r, s \in \mathbb{R}_+,$$

denotes the number of sell orders which have arrived up to time t *with relative price* p_r *and volume (size)* s*. We introduce its average value and set*

$$\alpha^s_\theta : \mathcal{B}\left([0,T] \times \mathbb{R}^2_+\right) \to \mathbb{R}$$

with

$$\alpha^s_\theta(C) = E\left(\mathcal{A}^s_\theta(C)\right), \qquad \text{for all } C \in \mathcal{B}\left([0,T] \times \mathbb{R}^2_+\right).$$

In the case where $\mathcal{A}^s_\theta$ *is a Poisson random measure, we call* α^s_θ ***intensity measure*** *of* $\mathcal{A}^s_\theta$*. If we look at the* ***extended arrival sell order book****, where we also take care of the absolute price, we will use the same notation* $\mathcal{A}^s_\theta$*, where now* $\mathcal{A}^s_\theta : \Omega \times \mathcal{B}\left([0,T] \times \mathbb{R}^3_+\right) \to \mathbb{R}$*. We note the following relation*

$$\mathcal{A}^s_\theta(A \times B \times C) = \mathcal{A}^s_\theta(A \times [0,\infty) \times B \times C), \qquad \text{for all } A \in \mathcal{B}([0,T]),\ B, C \in \mathcal{B}(\mathbb{R}_+).$$

Similar extended notations will be used for the other terms related to the arrival order book such as the intensity measure.

If we look at the **unconditional (extended) arrival sell order book**, we use the symbols $\mathcal{A}^s$ and α^s. The conditional (extended) arrival buy order book is denoted by $\mathcal{A}^b_\theta$ and its intensity measure by α^b_θ, the unconditional one by $\mathcal{A}^b$ and α^b.

Notation 2.8 (Order book) *We use the following notation for the* ***sell order book****, conditional on a realization of the stock price process up to time* T*,*

$$\mathcal{O}^s_\theta : \Omega \times [0,T] \times \mathcal{B}\left(\mathbb{R}^3_+\right) \to \mathbb{R}$$

where

$$\mathcal{O}^s_\theta\left(\{t\} \times [0,p_a] \times [0,p_r] \times [0,s]\right)$$

denotes the number of sell orders which are in the order book at time t with absolute price p_a, relative price p_r and volume (size) s ($p_a, p_r, s \in \mathbb{R}_+$). We introduce its expected value and set

$$\varpi_\theta^s : \mathcal{B}\left([0,T] \times \mathbb{R}_+^3\right) \to \mathbb{R}$$

where

$$\varpi_\theta^s(C) = E\left(\mathcal{O}_\theta^s(C)\right), \qquad \text{for all } C \in \mathcal{B}\left([0,T] \times \mathbb{R}_+^3\right).$$

In the case where $\mathcal{O}_\theta^s$ is a Poisson random measure, we call ϖ_θ^s ***intensity measure*** *of $\mathcal{O}_\theta^s$. If we look at the* ***unconditional sell order book,*** *we use the notation $\mathcal{O}^s$ and ϖ^s. The conditional arrival buy order book is denoted by $\mathcal{O}_\theta^b$ and its intensity measure by ϖ_θ^b, the unconditional one by $\mathcal{O}^b$ and ϖ^b. The* ***order book*** *itself is then defined as the difference between buy order book and sell order book and denoted by*

$$\mathcal{O} := \mathcal{O}^b - \mathcal{O}^s.$$

One of the main goals will be to find a characterization of the order book $\mathcal{O}$ based on suitable assumptions on the stock price process and the arrival process of orders.

2.3.2 The order book as a Poisson random measure

We start with the description of the arrival process of sell limit orders, conditional on a realization of the stock price process, denoted by $\{X_t(\omega)\}_{0\leqslant t\leqslant T} = \{x_t\}_{0\leqslant t\leqslant T}$ and usually indicated by θ (For θ, we omit the time parameter, it is always clear, which time horizon we consider.).

The arrival process is described as a random measure and denoted by $\mathcal{A}_\theta^s$:

$$\mathcal{A}_\theta^s : \Omega \times \mathcal{B}\left([0,T] \times \mathbb{R}_+^2\right) \to \mathbb{R}.$$

Assumption 2.9 *Assume that $\mathcal{A}_\theta^s$ is a Poisson random measure with finite intensity measure α_θ^s. This implies that*

1. *For almost all $\omega \in \Omega$, $\mathcal{A}_\theta^s(\omega;\cdot)$ is an integer-valued Radon measure on $[0,T] \times \mathbb{R}_+^2$: for any bounded measurable $A \subset [0,T] \times \mathbb{R}_+^2$, $\mathcal{A}_\theta^s(A) < \infty$ is an integer valued random variable.*

2. *For each measurable set $A \subset [0,T] \times \mathbb{R}_+^2$, $\mathcal{A}_\theta^s(\cdot, A) = \mathcal{A}_\theta^s(A)$ is a Poisson random variable with parameter $\alpha_\theta^s(A)$:*

$$\forall k \in \mathbb{N},\ P\left(\mathcal{A}_\theta^s(A) = k\right) = e^{-\alpha_\theta^s(A)} \frac{\left(\alpha_\theta^s(A)\right)^k}{k!}.$$

3. *For disjoint measurable sets $A_1, \ldots, A_n \subset [0,T] \times \mathbb{R}_+^2$, the variables $\mathcal{A}_\theta^s(A_1), \ldots, \mathcal{A}_\theta^s(A_n)$ are independent.*

In particular, $E\left(\mathcal{A}_\theta^s(A)\right) = \alpha_\theta^s(A)$ for all $A \in \mathcal{B}\left([0,T] \times \mathbb{R}_+^2\right)$.

Remark 2.10 *Our use of Poisson random measures can be justified by the following assumption: There is a large number of potential buyers and sellers in the market which are acting independently of each other. Each individual submits only occasionally an order to the exchange. We can therefore consider each order as originating from a different source, and hence unrelated to any other order.*

Using a Poisson random measure for the arrival order book has another nice feature. It is entirely described once we fix the intensity measure α_θ^s. In other words, we only have to find the average number of arriving orders in the set $A \in \mathcal{B}\left([0,T] \times \mathbb{R}_+^2\right)$ from empirical data. Let us give one example: What we observe in the market is e.g. the number of orders arriving at time t with relative price p. Denote them by $f_t(p)$. What does that mean for α_θ^s? This implies

$$\alpha_\theta^s(\{dt\}, [0,p], [0,\infty)) = \int_0^p f_s(\tilde{p})\ d\tilde{p}\ dt.$$

Basically, we have to observe the arrival times, relative prices and sizes of incoming orders. This gives us the input for the intensity measure α_θ^s. If one makes some independence assumptions, one could split the task and estimate those three properties separately. With this description of the arrival process we now have to go to the order book.

The path to the order book To reach the order book based on the arrival process, we need to keep track of the absolute prices. Therefore, we have to extend the random measure $\mathcal{A}_\theta^s$ on $\Omega \times \mathcal{B}\left([0,T] \times \mathbb{R}_+^2\right)$ to the space $\Omega \times \mathcal{B}\left([0,T] \times \mathbb{R}_+^3\right)$. The extended random measure will again be denoted by $\mathcal{A}_\theta^s$. We are confident that this will not lead to confusion. If $\mathcal{A}_\theta^s$ has four parameters, they are time, absolute price, relative price and size. If $\mathcal{A}_\theta^s$ has only three parameters, we do not consider the absolute price.

Definition 2.11 (extended arrival book) *We define the* ***extended sell arrival book*** *as*

$$\mathcal{A}_\theta^s : \Omega \times \mathcal{B}\left([0,T] \times \mathbb{R}_+^3\right) \rightarrow \mathbb{R}$$

with

$$\mathcal{A}_\theta^s(\omega, A) := \mathcal{A}_\theta^s\left(\omega, E_\theta^A\right)$$

where

$$E_\theta^A = \{(t,p,s) \mid (t,c,p,s) \in A \text{ such that } c = x_t + p\}$$

for all $A \in \mathcal{B}\left([0,T] \times \mathbb{R}_+^3\right)$, $\omega \in \Omega$.

Our first result is as follows:

Proposition 2.12 *Under Assumption* (2.9)*, the extended sell arrival book is also a Poisson random measure. Its intensity measure* α_θ^s *is given by*

$$\alpha_\theta^s(A) = \alpha_\theta^s\left(E_\theta^A\right)$$

for all $A \in \mathcal{B}\left([0,T] \times \mathbb{R}_+^3\right)$.

Proof. Observe that we include in the intensity measure only orders with an absolute price that is the sum of the corresponding relative price and the reference price. Let $A, B \in \mathcal{B}\left([0,T] \times \mathbb{R}_+^3\right)$ with $A \cap B = \emptyset$ and $i, j \in \mathbb{N}$. Then

$$\begin{aligned} & P\left(\mathcal{A}_\theta^s(A) = i \text{ and } \mathcal{A}_\theta^s(B) = j\right) = P\left(\mathcal{A}_\theta^s\left(E_\theta^A\right) = i \text{ and } \mathcal{A}_\theta^s\left(E_\theta^B\right) = j\right) \\ = \ & P\left(\mathcal{A}_\theta^s\left(E_\theta^A\right) = i\right) \cdot P\left(\mathcal{A}_\theta^s\left(E_\theta^B\right) = j\right) = P\left(\mathcal{A}_\theta^s(A) = i\right) \cdot P\left(\mathcal{A}_\theta^s(B) = j\right) \end{aligned}$$

by Assumption (2.9). This result implies that the extended sell arrival order book is again a Poisson random measure. ■

Lemma 2.13 (Some properties of α_θ^s) *Here we list a few straightforward properties of* α_θ^s :

a)

$$\alpha_\theta^s\left(\{t\}, [x_t, x_t + p], [0,p], [0,s]\right) = \alpha_\theta^s\left(\{t\}, [0,p], [0,s]\right),$$

b)

$$\begin{aligned} &\int_0^t \alpha_\theta^s\left(\{du\}, [x_u, x_u + p], [0,p], [0,s]\right) \\ = &\int_0^t \alpha_\theta^s\left(\{du\}, [0,p], [0,s]\right) = \alpha_\theta^s\left([0,t], [0,p], [0,s]\right), \end{aligned}$$

c)

$$\begin{aligned} &\alpha_\theta^s\left([0,t], \left[\min_{0\leqslant s\leqslant t} x_s + p, \max_{0\leqslant s\leqslant t} x_s + p\right], [0,p], [0,s]\right) \\ \geqslant\ &\alpha_\theta^s\left([0,t], [0,p], [0,s]\right) \\ \geqslant\ &\alpha_\theta^s\left([0,t], [x_u + p, x_u + p], [0,p], [0,s]\right) \end{aligned}$$

for all $u \in [0,t]$.

Proof. The proof is clear by using the definition of α_θ^s. ■

Now we need to go to the order book by always subtracting sell limit orders which are lower than the current price X. This gives us the sell order book $\mathcal{O}_\theta^s$. Also introduce the set D_T^s which denotes the complement of the area which has to be "cut off" at time T if the stock price is at X_T. We define the (random) set D_T^s as follows

$$D_T^s = \bigcup_{0\leqslant t\leqslant T} \left([0,t] \times \{X_t\} \times [0,\infty) \times [0,\infty)\right) \cup \{(t,p,p_r,s) \mid 0 \leqslant s \leqslant t\ ,\ p_r \geqslant 0\ ,\ p \geqslant X_s\}$$

and

$$D_T^{s,\theta} = \bigcup_{0\leqslant t\leqslant T} \left([0,t] \times \{x_t\} \times [0,\infty) \times [0,\infty)\right) \cup \{(t,p,p_r,s) \mid 0 \leqslant s \leqslant t\ ,\ p_r \geqslant 0\ ,\ p \geqslant x_s\}. \tag{2.3}$$

Intuitively, the set D_T^s denotes the complement of the area in which orders are executed up to time T.

Definition 2.14 (conditional sell order book) *The* ***conditional sell order book*** *is defined as the following mapping*

$$\begin{aligned} \mathcal{O}_\theta^s \quad &: \quad \Omega \times [0,T] \times \mathcal{B}\left(\mathbb{R}_+^3\right) \to \mathbb{N} \\ \mathcal{O}_\theta^s\left(\{t\}, A\right) \quad &\mapsto \quad \mathcal{A}_\theta^s\left(\left([0,t] \times A\right) \cap D_t^{s,\theta}\right) \end{aligned}$$

for all $A \in \mathcal{B}\left(\mathbb{R}_+^3\right)$ *and* $t \in [0,T]$ *where* $D_t^{s,\theta}$ *is given by* (2.3).

Equivalently, we could consider $\mathcal{O}_\theta^s$ as the mapping

$$\begin{aligned} \mathcal{O}_\theta^s \quad &: \quad \Omega \times [0,T] \to \left(\mathbb{M}\left(\mathbb{R}_+^3, \mathcal{B}\left(\mathbb{R}_+^3\right)\right),\ \mathcal{M}\left(\mathbb{R}_+^3, \mathcal{B}\left(\mathbb{R}_+^3\right)\right)\right) \\ \left(\mathcal{O}_\theta^s(\omega,t)\right)(A) \quad &\mapsto \quad \mathcal{A}_\theta^s\left(\left([0,t], A\right) \cap D_t^{s,\theta}\right) \end{aligned}$$

a random process on $[0,T]$ with integer-valued measures as outcome. The set $\mathbb{M}\left(\mathbb{R}_+^3,\ \mathcal{B}\left(\mathbb{R}_+^3\right)\right)$ denotes the set of all integer-valued random measures on $\mathbb{R}_+^3$; $\mathcal{M}\left(\mathbb{R}_+^3,\ \mathcal{B}\left(\mathbb{R}_+^3\right)\right)$ is the corresponding $\sigma-$algebra.

Theorem 2.15 *The conditional sell order book $\mathcal{O}_\theta^s : \Omega \times [0,T] \times \mathcal{B}\left(\mathbb{R}_+^3\right) \to \mathbb{N}$ is a Poisson process for every $t \in [0,T]$ with intensity measure ϖ_θ^s given by*

$$\varpi_\theta^s\left(\{t\} \times A\right) = \alpha_\theta^s\left(\left([0,t] \times A\right) \cap D_t^{s,\theta}\right)$$

for all $t \in [0,T]$ and $A \in \mathcal{B}\left(\mathbb{R}_+^3\right)$.

Proof. Let $t \in [0,T]$, $k \in \mathbb{N}$ and $A \in \mathcal{B}\left(\mathbb{R}_+^3\right)$. Then

$$\begin{aligned} & P\left(\mathcal{O}_\theta^s\left(\{t\} \times A\right) = k\right) \\ = \ & P\left(\mathcal{A}_\theta^s\left(\left([0,t] \times A\right) \cap D_t^{s,\theta}\right) = k\right) \\ = \ & \frac{\alpha_\theta^s\left(\left([0,t] \times A\right) \cap D_t^{s,\theta}\right)^k}{k!} e^{-\alpha_\theta^s\left(\left([0,t] \times A\right) \cap D_t^{s,\theta}\right)} \end{aligned}$$

This implies that $\mathcal{O}_\theta^s\left(\{t\} \times A\right)$ has a Poisson distribution with intensity measure

$$\alpha_\theta^s\left(\left([0,t] \times A\right) \cap D_t^{s,\theta}\right).$$

The claim follows by observing that for given $t \in [0,T]$ and $A, B \in \mathcal{B}\left(\mathbb{R}_+^3\right)$ with A, B disjoint, the random variables $\mathcal{O}_\theta^s\left(\{t\} \times A\right)$ and $\mathcal{O}_\theta^s\left(\{t\} \times B\right)$ are independent since $\mathcal{A}_\theta^s\left([0,t] \times C\right)$ is a Poisson point process for every $C \in \mathcal{B}\left(\mathbb{R}_+^3\right)$. ■

Definition 2.16 (sell order book) *We define the (unconditional)* ***sell order book*** *$\mathcal{O}^s : \Omega \times [0,T] \times \mathcal{B}\left(\mathbb{R}_+^3\right) \to \mathbb{N}$ as the mixture of the conditional sell order books $\mathcal{O}_\theta^s$ with respect to the distribution of the stock price process $(X_t)_{t \geqslant 0}$.*

Theorem 2.17 *Under Assumption (2.9), $\mathcal{O}^s$ is a mixed Poisson process for every $t \in [0,T]$. Its intensity measure ϖ^s is given as a mixture of the individual intensity measures of $\mathcal{O}_\theta^s$.*

Proof. The proof is clear from the definition of the sell order book and Theorem (2.15). ■

We finish with a property of mixed Poisson processes.

Lemma 2.18 *The distribution of $\mathcal{O}^s$ is uniquely defined by the distribution of its intensity measure ϖ^s.*

Proof. Use the description of a doubly stochastic Poisson process via its Laplace functional. ■

The order book In the previous sections we constructed the sell order book. The buy order book is set up analogously with the notation $\mathcal{O}^b$ for the random measure describing all buy limit orders and ϖ^b for its intensity measure. It turned out that both are Cox processes. (Note that in the literature one sometimes uses the term Cox processes only if the intensity measures of the Poisson processes conditional on θ are of the form $\nu(\cdot \mid \theta) = \theta\nu_0$ for some ν_0 fixed. In this case we would speak of a Cox process with multiplier ν_0. We use the term Cox process whenever we have a mixture of Poisson processes.) The last step to the order book is straightforward. Recall that buy limit orders are always lower than the stock price and sell limit orders higher. Therefore we define the order book $\mathcal{O}$ as

$$\mathcal{O} : \Omega \times [0, T] \times \mathcal{B}\left(\mathbb{R}_+^3\right) \to \mathbb{N}$$

with

$$\mathcal{O} := \mathcal{O}^b - \mathcal{O}^s. \tag{2.4}$$

We use the convention that sell orders appear with a negative sign in the order book. Summing up, we obtain the following result.

Theorem 2.19 *The order book $\mathcal{O}$ is the difference of two doubly stochastic Poisson processes for every $t \in [0, T]$ with intensity measure given by $\varpi^b - \varpi^s$.*

Proof. The proof is now just a combination of Theorems (2.4) and (2.17). ■

2.4 The sell order book as infinite sum of Bernoulli random variables

The set D_T^s is difficult to deal with. Recall that this set basically describes the complement of the area in which orders are executed. Here we describe another approach in order to derive a precise description of the intensity measure α_θ^s.

Assumptions and Notations Throughout this section, Assumption (2.9) will be valid. Set $E := [0, t] \times [0, \infty) \times [0, \infty)$. Then we can represent $\mathcal{A}_\theta^s$ as follows:

$$\mathcal{A}_\theta^s(A) = \sum_{i=1}^{\mathcal{A}_\theta^s(E)} 1_A(Y_i)$$

where the Y_i are *i.i.d.* random variables with distribution $P(Y_i \in A) = \frac{\alpha_\theta^s(A)}{\alpha_\theta^s(E)}$ for $A \in \mathcal{B}\left([0, t] \times \mathbb{R}_+^2\right)$ and $\mathcal{A}_\theta^s(E)$ is a Poisson random variable with mean $\alpha_\theta^s(E)$, independent of the Y_i $(i = 1, 2, \dots)$. We introduce $(T_i, R_i, S_i) := Y_i$, set $(T, R, S) := Y := Y_1$ and make the assumption that the random variable Y has a density with respect to P, denoted by $f_{(T,R,S)}$. Introduce the notations

$$m_\theta(t, p, \infty, s) := \int_0^t \int_{\max_{u \leqslant s \leqslant t} x_s - x_u}^{p - x_u} \int_0^s f_{(T,R,S)}(u, v, w) \; dw \; dv \; du$$

and

$$m(t, p, \infty, s) := \int_0^t \int_{\max_{u \leqslant s \leqslant t} X_s - X_u}^{p - X_u} \int_0^s f_{(T,R,S)}(u, v, w) \; dw \; dv \; du$$

$(p, s \in \bar{\mathbb{R}}_+)$. We use the shortcut

$$m_\theta^s := m_\theta(t, p, \infty, \infty) = \int_0^t \int_{\max_{u \leqslant s \leqslant t} x_s - x_u}^{p - x_u} \int_0^\infty f_{(T,R,S)}(u, v, w)\ dw\ dv\ du.$$

For all integrals above, we use the convention that $\int_a^b f(x)\ dx = 0$ whenever $b \leqslant a$.

For $i = 1, 2, \ldots$ we order the random variables T_i in increasing order and denote the ordered sequence by $\tilde{T}_i$. Clearly, the distributions of $\tilde{T}_i$ can be easily calculated from the distributions of T_i (i−th order statistics). We look at the measure $\mathcal{A}_\theta^s$ and give a representation of the number of orders with absolute price in $[0, p]$, arbitrary relative price and size in $[0, s]$, $s, p \in \mathbb{R}_+$.

Theorem 2.20

$$\begin{aligned}
& \mathcal{O}_\theta^s(\{t\}, [0,p], [0,\infty), [0,s]) \\
= \ & \mathcal{A}_\theta^s\left(\left[0, \tilde{T}_1\right], \left[\max_{\tilde{T}_1 \leqslant s \leqslant t} x_s - x_{\tilde{T}_1}, p - x_{\tilde{T}_1}\right], [0,s]\right) 1_{\{\tilde{T}_1 \leqslant t\}} \\
& + \mathcal{A}_\theta^s\left(\left(\tilde{T}_1, \tilde{T}_2\right], \left[\max_{\tilde{T}_2 \leqslant s \leqslant t} x_s - x_{\tilde{T}_2}, p - x_{\tilde{T}_2}\right], [0,s]\right) 1_{\{\tilde{T}_2 \leqslant t\}} \\
& + \ \ldots \\
= \ & \sum_{k=0}^{\infty} \mathcal{A}_\theta^s\left(\left(\tilde{T}_k, \tilde{T}_{k+1}\right], \left[\max_{\tilde{T}_{k+1} \leqslant s \leqslant t} x_s - x_{\tilde{T}_{k+1}}, p - x_{\tilde{T}_{k+1}}\right], [0,s]\right) 1_{\{\tilde{T}_{k+1} \leqslant t\}}.
\end{aligned}$$

Each term $\mathcal{A}_\theta^s\left(\left(\tilde{T}_k, \tilde{T}_{k+1}\right], \left[\max_{\tilde{T}_{k+1} \leqslant s \leqslant t} x_s - x_{\tilde{T}_{k+1}}, p - x_{\tilde{T}_{k+1}}\right], [0,s]\right) 1_{\{\tilde{T}_{k+1} \leqslant t\}}$ *has a Bernoulli distribution.*

Proof. Observe that a limit order with absolute price equal to $p_a \leqslant p$, submitted at $\tilde{T}_k$ is still in the order book at time t if $\max_{\tilde{T}_k \leqslant s \leqslant t} x_s \leqslant p_a \leqslant p$. The infinite sum above is well-defined since, by the assumptions made in this section here, $\mathcal{A}_\theta^s$ has a finite intensity measure and we add up only finitely many terms P−a.s. ■

Theorem 2.21 *Recall that* $E = [0,t] \times [0,\infty) \times [0,\infty)$. *The intensity measure of* $\mathcal{O}_\theta^s$ *is given by*

$$\begin{aligned}
\varpi_\theta^s(\{t\}, [0,p], [0,\infty), [0,s]) &= E\left(\mathcal{O}_\theta^s(\{t\}, [0,p], [0,\infty), [0,s])\right) \\
&= m_\theta(t, p, \infty, s) \cdot \alpha_\theta^s(E)
\end{aligned}$$

$(p, s \in \mathbb{R}_+)$.

Remark 2.22 *Observe that* $\alpha_\theta^s(E)$ *just denotes the total expected number of orders which arrive up to time* t. *Of course, only a fraction of them will be in the order book at time* t, *which is exactly expressed by* $m_\theta(t, p, \infty, s)$. *In Theorem* (2.15), *it was shown that* $\mathcal{O}_\theta^s(\{t\} \times A)$ *has a Poisson distribution for every* $t \in [0, T]$ *and all* $A \in \mathcal{B}(\mathbb{R}_+^3)$. *Together with this theorem here, we now have a precise description of the conditional sell order book* $\mathcal{O}_\theta^s$. *An extension of the above theorem to arbitrary measurable sets is possible.*

Proof. The random variable

$$\mathcal{A}_\theta^s\left(\left(\tilde{T}_k,\tilde{T}_{k+1}\right],\left[\max_{\tilde{T}_{k+1}\leqslant s\leqslant t} x_s - x_{\tilde{T}_{k+1}}, p - x_{\tilde{T}_{k+1}}\right],[0,s]\right)$$

can take either the value 0 or 1. We can write

$$\mathcal{A}_\theta^s(A) = \sum_{i=1}^{\mathcal{A}_\theta^s(E)} 1_A(Y_i), \qquad A \in \mathcal{B}\left([0,t]\times\mathbb{R}_+^2\right),$$

where the random variables $Y_i = (T_i, R_i, S_i)$ are i.i.d., independent of $\mathcal{A}_\theta^s(E)$ and $P(Y_i \in A) = \frac{\alpha_\theta^s(A)}{\alpha_\theta^s(E)}$. We first calculate

$$\begin{aligned} & P\left(\mathcal{A}_\theta^s\left(\left(0,\tilde{T}_1\right],\left[\max_{\tilde{T}_1\leqslant s\leqslant t} x_s - x_{\tilde{T}_1}, p - x_{\tilde{T}_1}\right],[0,s]\right)1_{\{\tilde{T}_1\leqslant t\}} = 1\right) \\ = & \sum_{m=0}^{\infty} P\left(\sum_{i=1}^{m} 1_{\{T_i,R_i,S_i\}}\left(\left(0,\tilde{T}_1\right],\left[\max_{\tilde{T}_1\leqslant s\leqslant t} x_s - x_{\tilde{T}_1}, p - x_{\tilde{T}_1}\right],[0,s]\right) = 1,\ \tilde{T}_1 \leqslant t \mid \mathcal{A}_\theta^s(E) = m\right) \\ & \times P\left(\mathcal{A}_\theta^s(E) = m\right). \end{aligned}$$

Now we look at each individual term:

$$\begin{aligned} & P\left(\sum_{i=1}^{m} 1_{\{T_i,R_i,S_i\}}\left(\left(0,\tilde{T}_1\right],\left[\max_{\tilde{T}_1\leqslant s\leqslant t} x_s - x_{\tilde{T}_1}, p - x_{\tilde{T}_1}\right],[0,s]\right) = 1,\ \tilde{T}_1 \leqslant t \mid \mathcal{A}_\theta^s(E) = m\right) \\ = & P\left(R_1 \in \left[\max_{T_1\leqslant s\leqslant t} x_s - x_{T_1}, p - x_{T_1}\right] \text{ and } T_1 \leqslant t \text{ and } S_1 \leqslant s\right)\frac{1}{m} \\ & + P\left(R_2 \in \left[\max_{T_2\leqslant s\leqslant t} x_s - x_{T_2}, p - x_{T_2}\right] \text{ and } T_2 \leqslant t \text{ and } S_2 \leqslant s\right)\frac{1}{m} \\ & + \dots \\ & + P\left(R_m \in \left[\max_{T_m\leqslant s\leqslant t} x_m - x_{T_m}, p - x_{T_m}\right] \text{ and } T_m \leqslant t \text{ and } S_m \leqslant s\right)\frac{1}{m}. \end{aligned}$$

From this we obtain:

$$\begin{aligned} & P\left(R_1 \in \left[\max_{T_1\leqslant s\leqslant t} x_s - x_{T_1}, p - x_{T_1}\right] \text{ and } T_1 \leqslant t \text{ and } S_1 \leqslant s\right) \\ = & P\left(R_1 \leqslant p - x_{T_1} \text{ and } R_1 \geqslant \max_{T_1\leqslant s\leqslant t} x_s - x_{T_1} \text{ and } T_1 \leqslant t \text{ and } S_1 \leqslant s\right) \\ = & \int_0^t \int_{\max_{u\leqslant s\leqslant t} x_s - x_u}^{p-x_u} \int_0^s f_{(T_1,R_1,S_1)}(u,v,w)\ dw\ dv\ du \\ = & \int_0^t \int_{\max_{u\leqslant s\leqslant t} x_s - x_u}^{p-x_u} \int_0^s f_{(T,R,S)}(u,v,w)\ dw\ dv\ du = m_\theta(t,p,\infty,s) \end{aligned}$$

where $f_{(T,R,S)}$ is the density of $Y = (T,R,S)$. Since the processes (T_i, R_i, S_i) are i.i.d. we get

$$\begin{aligned} & P\left(\sum_{i=1}^{m} 1_{\{T_i,R_i,S_i\}}\left(\left(0,\tilde{T}_1\right],\left[\max_{\tilde{T}_1\leqslant s\leqslant t} x_s - x_{\tilde{T}_1}, p - x_{\tilde{T}_1}\right],[0,s]\right) = 1,\ \tilde{T}_1 \leqslant t \mid \mathcal{A}_\theta^s(E) = m\right) \\ = & m_\theta(t,p,\infty,s). \end{aligned}$$

Then

$$\begin{aligned}
& P\left(\mathcal{A}_\theta^s\left(\left(0,\tilde{T}_1\right],\left[\max_{\tilde{T}_1\leqslant s\leqslant t} x_s - x_{\tilde{T}_1}, p - x_{\tilde{T}_1}\right],[0,s]\right)1_{\{\tilde{T}_1\leqslant t\}} = 1\right) \\
= & \sum_{m=1}^{\infty} m_\theta\left(t,p,\infty,s\right)\cdot P\left(\mathcal{A}_\theta^s\left(E\right)=m\right).
\end{aligned}$$

For the remaining terms:

$$\begin{aligned}
& P\left(\mathcal{A}_\theta^s\left(\left(\tilde{T}_k,\tilde{T}_{k+1}\right],\left[\max_{\tilde{T}_{k+1}\leqslant s\leqslant t} x_s - x_{\tilde{T}_{k+1}}, p - x_{\tilde{T}_1}\right],[0,s]\right)1_{\{\tilde{T}_{k+1}\leqslant t\}} = 1\right) \\
= & \; m_\theta\left(t,p,\infty,s\right)\sum_{m=k+1}^{\infty} P\left(\mathcal{A}_\theta^s\left(E\right)=m\right)
\end{aligned}$$

since for $k \leqslant m$:

$$P\left(\sum_{i=1}^{m} 1_{\{T_i,R_i,S_i\}}\left(\left(\tilde{T}_k,\tilde{T}_{k+1}\right],\left[\max_{\tilde{T}_{k+1}\leqslant s\leqslant t} x_s - x_{\tilde{T}_{k+1}}, p - x_{\tilde{T}_1}\right],[0,s]\right) = 1,\ \tilde{T}_{k+1}\leqslant t\right) = 0.$$

Based on this, we can calculate

$$\begin{aligned}
& E\left(\mathcal{O}_\theta^s\left(\{t\},[0,p],[0,\infty),[0,s]\right)\right) \\
= & \sum_{k=0}^{\infty} E\left(\mathcal{A}_\theta^s\left(\left(\tilde{T}_k,\tilde{T}_{k+1}\right],\left[\max_{\tilde{T}_{k+1}\leqslant s\leqslant t} x_s - x_{\tilde{T}_{k+1}}, p - x_{\tilde{T}_1}\right],[0,s]\right)1_{\{\tilde{T}_{k+1}\leqslant t\}}\right) \\
= & \; m_\theta\left(t,p,\infty,s\right)\cdot\sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty} P\left(\mathcal{A}_\theta^s\left(E\right)=m\right) \\
= & \; m_\theta\left(t,p,\infty,s\right)\cdot\sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty} \frac{\alpha_\theta^s\left(E\right)^m}{m!}e^{-\alpha_\theta^s(E)} \\
= & \; m_\theta\left(t,p,\infty,s\right)\cdot\alpha_\theta^s\left(E\right).
\end{aligned}$$

■

Therefore it remains to compute

$$\int_0^t \int_{\max_{u\leqslant s\leqslant t} x_s - x_u}^{p-x_u} \int_0^s f_{(T,R,S)}\left(u,v,w\right)\ dw\ dv\ du = m_\theta\left(t,p,\infty,s\right).$$

This is a task which can only be achieved using numerical simulations, given a chosen density $f_{(T,R,S)}$.

From the above theorem, we can quickly deduce several other relevant quantities. We just mention two of them, which seem to be useful in practice.

Corollary 2.23 *We obtain*

$$\varpi_\theta^s\left(\{t\},[0,p],[0,\infty),[0,\infty)\right) = m_\theta^s\cdot\alpha_\theta^s\left(E\right)$$

($p \in \mathbb{R}_+$) for the average number of sell orders in the book at time t with absolute price between 0 and p. Furthermore, we get

$$\varpi_\theta^s(\{t\}, [0,\infty), [0,p], [0,\infty)) = m_\theta(t,\infty,p,\infty) \cdot \alpha_\theta^s(E)$$

for the average number of sell orders in the book at time t with relative price between 0 and p, where

$$m_\theta(t,\infty,p,\infty) := \int_0^t \int_{\max_{u \leqslant s \leqslant t} x_s - x_u}^p \int_0^\infty f_{(T,R,S)}(u,v,w)\ dw\ dv\ du.$$

Proof. A slight modification of the previous proof leads to this corollary. ∎

Theorem 2.24 (Order book) *The order book can be characterized as the difference of the mixture of two infinite sums of Bernoulli distributed random variables, where the mixture is with respect to the distribution of the stock price.*

Proof. Combine Theorems (2.20) and (2.21) with (2.4). ∎

3 Applications and extensions

Here we consider several applications and extension of our model for the limit order book. Our model is build in a such way that we do not model the evolution of the bid and ask price, since we have only one transaction price. This might be a disadvantage of our proposed approach, however we will show that we can derive the bid and ask prices and we are even able to fit them to empirical observations. We consider the relation between the volume of orders in the book and the volatility of the stock price. Furthermore, a new type of options occurs naturally, a reverse Asian fixed strike lookback option. We also look at certain invariance properties of the distribution of the relative limit order price and characterize the waiting time until cancellation or execution. In the appendix a description both of the limit distribution of the order book and its maximum in the long run are given as well as a short investigation of the volume-volatility relation is undertaken.

3.1 The bid-ask spread

Here we have a look at the best bid and ask prices and the bid-ask spread.

Definition 3.1 *Define the **best ask price** a_t at time t as*

$$a_t = \inf\{p \mid \mathcal{O}_\theta^s(t,[0,p],[0,\infty),[0,\infty)) \geqslant 1\}$$

*and the **best bid price** b_t as*

$$b_t = \sup\{p \mid \mathcal{O}_\theta^b(t,[p,\infty),[0,\infty),[0,\infty)) \geqslant 1\}$$

*Since we can have an empty order book, we have to define an ask price for that case as well; we set $a_t = \infty$ if $\{\mathcal{O}_\theta^s(t,[0,p],[0,\infty),[0,\infty)) \geqslant 1\} = \emptyset$ and $b_t = 0$ if $\{\mathcal{O}_\theta^b(t,[p,\infty),[0,\infty),[0,\infty))\} = \emptyset$. The **bid-ask spread** is defined as $b_t - a_t$ provided that $a_t < \infty$ and $b_t > 0$.*

Set $E = [0,t] \times [0,\infty) \times [0,\infty)$ and define

$$\begin{aligned} m_\theta^s &= \int_0^t \int_{\max_{u \leqslant s \leqslant t} x_s - x_u}^{p - x_u} f_{(T,R)}(u,v)\ dv\ du, \\ m_\theta^b &= \int_0^t \int_{x_u - \min_{u \leqslant s \leqslant t} x_s}^{x_u - p} f_{(T^b,R^b)}(u,v)\ dv\ du \end{aligned}$$

, where $f_{(T,R)}$ is the joint density of (sell) order arrival time and relative price and $f_{(T^b,R^b)}$ denotes the joint density or order arrival time and relative price on the buy side which corresponds to the density $f_{(T,R)}$ on the sell side.

Lemma 3.2 *The distribution of the ask price is given by*

$$P(a_t \leqslant p) = \begin{cases} 1 - \exp\{-m_\theta^s \cdot \alpha_\theta^s(E)\} & \text{for } p \in \mathbb{R} \\ \exp\left\{-\alpha_\theta^s(E) \cdot \int_0^t \int_{\max_{u \leqslant s \leqslant t} X_s - X_u}^{\infty} f_{(T,R)}(u,v)\ dv\ du\right\} & \text{for } p = \infty \end{cases}.$$

Remark 3.3 *We see a liquidity effect here: If the total number of submitted orders $\alpha_\theta^s(E)$ increases, letting everything else constant, then the probability of the ask price being less than p increases. The probability $P(a_t \leqslant p)$ is a concave function of $\alpha_\theta^s(E)$.*

Proof. Observe that

$$\{a_t \leqslant p\} = \{\mathcal{O}_\theta^s(t,[0,p],[0,\infty),[0,\infty)) \geqslant 1\}.$$

Then

$$\begin{aligned} P(a_t \leqslant p) &= P(\mathcal{O}_\theta^s(t,[0,p],[0,\infty),[0,\infty)) \geqslant 1) \\ &= 1 - P(\mathcal{O}_\theta^s(t,[0,p],[0,\infty),[0,\infty)) = 0) \\ &= 1 - \exp\{-m_\theta^s \cdot \alpha_\theta^s(E)\}. \end{aligned}$$

The term $\alpha_\theta^s(E)$ does not depend on p. The term

$$m_\theta^s = \int_0^t \int_{\max_{u \leqslant s \leqslant t} x_s - x_u}^{p - x_u} f_{(T,R)}(u,v)\ dv\ du$$

has the following properties: for $p = x_t$ we get $m_\theta^s = 0$ as desired. Therefore

$$P(a_t \leqslant x_t) = 0$$

(which is obviously clear). Furthermore,

$$\lim_{p \to \infty} \int_0^t \int_{\max_{u \leqslant s \leqslant t} X_s - X_u}^{p - X_u} f_{(T,R)}(u,v)\ dv\ du = \int_0^t \int_{\max_{u \leqslant s \leqslant t} X_s - X_u}^{\infty} f_{(T,R)}(u,v)\ dv\ du.$$

Therefore

$$\begin{aligned} &\lim_{p \to \infty} P(a_t \leqslant p) \\ &= 1 - \exp\left\{-\alpha_\theta^s(E) \cdot \int_0^t \int_{\max_{u \leqslant s \leqslant t} X_s - X_u}^{\infty} f_{(T,R)}(u,v)\ dv\ du\right\} \end{aligned}$$

which is also intuitively clear, since there is a positive probability of having no sell order in the book. This probability is exactly given by

$$\exp\left\{-\alpha_\theta^s(E)\cdot\int_0^t\int_{\max_{u\leqslant s\leqslant t}X_s-X_u}^{\infty}f_{(T,R)}(u,v)\ dv\ du\right\}.$$

■

Remark 3.4 *By reasonably choosing $\alpha_\theta^s(E)$ we can make the probability of the ask price being less than p as small as we like. The quantity $\alpha_\theta^s(E)$ therefore also gives us one possibility to adjust our bid-ask spread to empirically observed values. Clearly, there is also a liquidity effect: If we have a large total number of orders ($\alpha_\theta^s(E)$ large), then the bid-ask spread will on average be lower, which coincides with empirical observations. Furthermore, if m_θ^s is smaller, i.e. e.g. $\max_{u\leqslant s\leqslant t}X_s-X_u$ is smaller, then $P(a_t\leqslant p)$ is larger. (Usually, one says that a lower volatility leads to a smaller bid-ask spread. Here we see that it is not so much the volatility which influences the bid-ask spread but the quantity $\max_{u\leqslant s\leqslant t}X_s-X_u$. We see that it is more the future maximum deviation from the current price X_u which plays an important role.) The probability distribution of the ask price, conditional on a non-empty book, can easily be calculated from the previous lemma.*

Lemma 3.5 *Assume that the conditional sell order book $\mathcal{O}_\theta^s$ and the conditional buy order book $\mathcal{O}_\theta^b$ are independent. The distribution of the bid-ask spread at time t is given by*

$$\begin{aligned}
&P(a_t-b_t\leqslant p)\\
=\ &\frac{1}{\Psi}\int_0^p\int_0^\infty\left[\exp\left\{-\int_0^t\int_{x_u-\min_{u\leqslant s\leqslant t}x_s}^{x_u+y-x}f_{(T^b,R^b)}(u,v)\ dv\ du\cdot\alpha_\theta^b(E)\right\}\cdot\alpha_\theta^b(E)\right.\\
&\times\left(\int_0^t f_{(T^b,R^b)}(u,x_u+y-x)\ du\right)\\
&\times\exp\left\{-\int_0^t\int_{\max_{u\leqslant s\leqslant t}x_s-x_u}^{x-x_u}f_{(T,R)}(u,v)\ dv\ du\cdot\alpha_\theta^s(E)\right\}\\
&\left.\times\ \alpha_\theta^s(E)\cdot\int_0^t f_{(T,R)}(u,x-x_u)\ du\right]\ dx\ dy,
\end{aligned}$$

where

$$\begin{aligned}
\Psi\ &=\ P(a_t<\infty\ \text{and}\ b_t>0)\\
&=\ \left(1-\exp\left\{-\alpha_\theta^s(E)\cdot\int_0^t\int_{\max_{u\leqslant s\leqslant t}X_s-X_u}^{\infty}f_{(T,R)}(u,v)\ dv\ du\right\}\right)\\
&\quad\times\left(1-\exp\left\{-\int_0^t\int_{x_u-\min_{u\leqslant s\leqslant t}x_s}^{x_u}f_{(T^b,R^b)}(u,v)\ dv\ du\cdot\alpha_\theta^b(E)\right\}\right)
\end{aligned}$$

Proof. We calculate the conditional densities of the bid and the ask price:

$$\begin{aligned} & P\left(b_t \in dx \mid b_t > 0\right) \\ = & \exp\left\{-\int_0^t \int_{x_u - \min_{u \leqslant s \leqslant t} x_s}^{x_u - x} f_{\left(T^b, R^b\right)}(u, v)\ dv\ du \cdot \alpha_\theta^b(E)\right\} \cdot \alpha_\theta^b(E) \\ & \times \int_0^t f_{\left(T^b, R^b\right)}(u, x_u - x)\ du\ dx \end{aligned}$$

and

$$\begin{aligned} & P\left(a_t \in dx \mid a_t < \infty\right) \\ = & \exp\left\{-\int_0^t \int_{\max_{u \leqslant s \leqslant t} x_s - x_u}^{x - x_u} f_{(T,R)}(u, v)\ dv\ du \cdot \alpha_\theta^s(E)\right\} \cdot \alpha_\theta^s(E) \\ & \times \int_0^t f_{(T,R)}(u, x - x_u)\ du\ dx \end{aligned}$$

($x \in \mathbb{R}_+$). From this we compute the distribution of the bid-ask spread $a_t - b_t$ at time t. Using the convolution formula leads to

$$\begin{aligned} & P\left(a_t - b_t \leqslant p \mid b_t > 0 \text{ and } a_t < \infty\right) \\ = & \int_0^p \int_0^\infty \exp\left\{-\int_0^t \int_{x_u - \min_{u \leqslant s \leqslant t} x_s}^{x_u + y - x} f_{\left(T^b, R^b\right)}(u, v)\ dv\ du \cdot \alpha_\theta^b(E)\right\} \cdot \alpha_\theta^b(E) \\ & \times \int_0^t f_{\left(T^b, R^b\right)}(u, x_u + y - x)\ du \cdot \exp\left\{-\int_0^t \int_{\max_{u \leqslant s \leqslant t} x_s - x_u}^{x - x_u} f_{(T,R)}(u, v)\ dv\ du \cdot \alpha_\theta^s(E)\right\} \\ & \times \alpha_\theta^s(E) \cdot \int_0^t f_{(T,R)}(u, x - x_u)\ du\ dx\ dy. \end{aligned}$$

Note that $P(b_t > 0) = \left(1 - \exp\left\{-\int_0^t \int_{x_u - \min_{u \leqslant s \leqslant t} x_s}^{x_u} f_{\left(T^b, R^b\right)}(u, v)\ dv\ du \cdot \alpha_\theta^b(E)\right\}\right)$. ■

What is the effect of the total number of submitted orders? For this, we compute

$$\begin{aligned} & P\left(a_t - b_t \geqslant p\right) \\ = & \int_0^{x_t} \exp\left\{-\int_0^t \int_{\max_{u \leqslant s \leqslant t} x_s - x_u}^{p + x - x_u} f_{(T,R)}(u, v)\ dv\ du \cdot \alpha_\theta^s(E)\right\} \\ & \times \exp\left\{-\int_0^t \int_{x_u - \min_{u \leqslant s \leqslant t} x_s}^{x_u - x} f_{\left(T^b, R^b\right)}(u, v)\ dv\ du \cdot \alpha_\theta^b(E)\right\} \\ & \times \alpha_\theta^b(E) \cdot \int_0^t f_{\left(T^b, R^b\right)}(u, x_u - x)\ du\ dx \end{aligned}$$

Here we immediately see how an increase in $\alpha_\theta^s(E)$ affects the bid-ask spread:

$$\begin{aligned}
&\frac{d}{d\alpha_\theta^s(E)}P(a_t-b_t\geqslant p)\\
=&\frac{d}{d\alpha_\theta^s(E)}\int_0^{x_t}\exp\left\{-\int_0^t\int_{\max_{u\leqslant s\leqslant t}x_s-x_u}^{p+x-x_u} f_{(T,R)}(u,v)\ dv\ du\cdot\alpha_\theta^s(E)\right\}\\
&\times\exp\left\{-\int_0^t\int_{x_u-\min_{u\leqslant s\leqslant t}x_s}^{x_u-x} f_{(T^b,R^b)}(u,v)\ dv\ du\cdot\alpha_\theta^b(E)\right\}\\
&\times\alpha_\theta^b(E)\cdot\int_0^t f_{(T^b,R^b)}(u,x_u-p)\ du\ dx\\
=&\int_0^{x_t}-\exp\left\{-\int_0^t\int_{\max_{u\leqslant s\leqslant t}x_s-x_u}^{p+x-x_u} f_{(T,R)}(u,v)\ dv\ du\cdot\alpha_\theta^s(E)\right\}\cdot\vartheta\ dx
\end{aligned}$$

where

$$\begin{aligned}
\vartheta=&\exp\left\{-\int_0^t\int_{x_u-\min_{u\leqslant s\leqslant t}x_s}^{x_u-x} f_{(T^b,R^b)}(u,v)\ dv\ du\cdot\alpha_\theta^b(E)\right\}\cdot\alpha_\theta^b(E)\\
&\times\int_0^t f_{(T^b,R^b)}(u,x_u-p)\ du\cdot\int_0^t\int_{\max_{u\leqslant s\leqslant t}x_s-x_u}^{p+x-x_u} f_{(T,R)}(u,v)\ dv\ du.
\end{aligned}$$

Remark 3.6 *This shows that an increase in the volume of submitted sell orders leads to an exponential type of increase in the probability that the bid-ask spread stays below a given value p. Similarly, we can consider the effect of $\alpha_\theta^b(E)$. This is an effect which is usually reported in the literature. This result does not a priori contradict the findings in [PGS05], where the authors report that an increase in the total volume of traded orders increases the bid-ask spread. Our statement looks at the volume of submitted orders. In addition, we consider here a small investor model, whereas they explain their findings with the effect that large order submission could be directional. This is however only possible if we have informed traders or a large trader effect. Second, the market they investigate does have market makers. Therefore, during periods of large demand or supply, we expect the bid-ask spread to be large, since the market maker increases the spread to compensate for the additional risk. However, we do not have market makers which would incur such costs for bearing the additional risk. Note that we are still dealing with a small-investor model here. If, however, we include large or informed traders, we can have directional trades which would imply that the bid-ask spread could actually increase due to increasing volume.*

3.1.1 The heavy-tailedness of the bid-ask spread

We look at the heavy-tailedness of the distribution of the bid-ask spread. If the distribution of the best ask price a_t is heavy-tailed at time t, then the bid-ask spread is also heavy-tailed. So we are left with the calculation of the heavy-tailedness of the distribution of a_t. It turns out that under very mild conditions, the best ask and bid prices as well as the bid-ask spread are heavy-tailed. For the convenience of the reader, we recall the definition of heavy-tailed distributions.

Definition 3.7 (Heavy-tailed distributions) *A random variable with distribution F is said to be* ***heavy-tailed*** *if $\bar{F}(x) > 0, x \geqslant 0$, and*

$$\lim_{x\to\infty} P(X > x+y \,|X > x) = \lim_{x\to\infty} \frac{\bar{F}(x+y)}{\bar{F}(x)} = 1 \qquad \text{for all } y \geqslant 0.$$

A standard example is the Pareto distribution function with tail $\bar{F}(x) = x^{-\alpha}$, $x \geqslant 1$, where $\alpha > 0$ is a parameter.

We use

$$P(a_t \leqslant p) = 1 - \exp\{-f(p) \cdot \alpha_\theta^s(E)\}$$

where $f(p) := \int_0^t \int_{\max_{u\leqslant s\leqslant t} x_s - x_u}^{p-x_u} f_{(T,R)}(u,v)\ dv\ du$. Which choices of functions $f(p)$ lead to a heavy-tailed distribution for the ask price?

Lemma 3.8 *Assume that $f(p) < f(\infty)$ for all $p \in \mathbb{R}_+$. Then the best ask price has a heavy-tailed distribution.*

Remark 3.9 *The assumption in the previous lemma is e.g. satisfied if $f_{(T,R)}(u,v) > 0$ for all $u, v \in \mathbb{R}$.*

Proof. By assumption, we have that $P(a_t \geqslant p) > 0$. Using the definition of heavy-tailedness, we compute

$$\begin{aligned}
& \lim_{p\to\infty} \frac{P(a_t \geqslant p+z)}{P(a_t \geqslant p)} \\
= \quad & \lim_{p\to\infty} \frac{\exp\{-f(p+z)\cdot\alpha_\theta^s(E)\} - \exp\{-f(\infty)\cdot\alpha_\theta^s(E)\}}{\exp\{-f(p)\cdot\alpha_\theta^s(E)\} - \exp\{-f(\infty)\cdot\alpha_\theta^s(E)\}} = 1 \\
\iff \quad & \lim_{p\to\infty} \exp\{(f(p) - f(p+z))\cdot\alpha_\theta^s(E)\} = 1 \\
\iff \quad & \lim_{p\to\infty} (f(p) - f(p+z)) = 0
\end{aligned}$$

Since

$$f(p) = \int_0^t \int_{\max_{u\leqslant s\leqslant t} x_s - x_u}^{p-x_u} f_{(T,R)}(u,v)\ dv\ du$$

we get

$$\begin{aligned}
& f(p+z) - f(p) \\
= \quad & \int_0^t \int_{\max_{u\leqslant s\leqslant t} x_s - x_u}^{p+z-x_u} f_{(T,R)}(u,v)\ dv\ du - \int_0^t \int_{\max_{u\leqslant s\leqslant t} x_s - x_u}^{p-x_u} f_{(T,R)}(u,v)\ dv\ du \\
= \quad & \int_0^t \int_{p-x_u}^{p+z-x_u} f_{(T,R)}(u,v)\ dv\ du
\end{aligned}$$

Then

$$\lim_{p\to\infty} f(p+z) - f(p) = 0$$

i.e. the ask price is always heavy-tailed. ■

From this we get the heavy-tailedness of the bid-ask spread with the help of the following lemma:

Lemma 3.10 *If X is heavy-tailed and Y is any nonnegative random variable, that is independent of X, then $X - Y$ is also heavy-tailed.*

Proof. Use a basic application of the bounded convergence theorem. ■

We finish with the main result.

Theorem 3.11 *Let $t \geqslant 0$. Assume that a_t and b_t are independent and $f(p) < f(\infty)$ for all $p \in \mathbb{R}_+$. Then the bid-ask spread has a heavy-tailed distribution.*

Proof. This is now just a combination of Lemmata (3.8) and (3.10). ■

3.1.2 Empty sell order book up to price p

We can also calculate the probability, that there is no demand on the sell side with a limit price lower than p:

$$\begin{aligned} & P\left(a_t \geqslant p \text{ or } a_t = \infty\right) \\ = & \exp\left\{-\int_0^t \int_{\max_{u\leqslant s\leqslant t} x_s - x_u}^{p-x_u} f_{(T,R)}\left(u,v\right)\ dv\ du \cdot \alpha_\theta^s\left(E\right)\right\}. \end{aligned}$$

From this, we can immediately calculate the minimum impact of a large buy market order: The probability that the large trader has to pay at least the price p (i.e. the price impact is at least $p - x_t$) is given by

$$\exp\left\{-f\left(p\right)\cdot\alpha_\theta^s\left(E\right)\right\}.$$

Similar arguments apply to the bid side of the limit order book.

3.2 The intensity measure as an exotic option - A reverse Asian fixed strike lookback option

This section gives a brief flavour of one of the topics which arise in the context of this model. In a particular case, we obtain a new type of exotic options. If we have a uniform density $f_{(T,R)}$ on $[0,T]\times[0,P]$, then

$$\begin{aligned} & P \cdot \int_0^T \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{P-X_u} f_{(T,R)}\left(u,v\right)\ dv\ du \\ = & \frac{1}{T}\int_0^T \max\left\{\left(P - \max_{u\leqslant s\leqslant T} X_s\right), 0\right\}\ du. \end{aligned}$$

We buy an option which has the following payoff: Fix u. Assume that $P \geqslant \max_{u\leqslant s\leqslant t} X_s$. At time u, we get P (imagine we sell one stock at price P) and have to pay $\max_{u\leqslant s\leqslant T} X_u$. In words, we have to pay the average of the remaining maximum of the stock price, provided it is less than P. We call this a reverse Asian fixed strike lookback option. Imagine we are at time T. Looking back, we are allowed to sell our stock at the limit price P at time u, provided the share price stays below P for the remaining period. Then we have to pay the maximum share price. However, we do not have to decide when to do this transaction: We always get the average of $\max\left\{\left(P - \max_{\cdot\leqslant s\leqslant T} X_s\right), 0\right\}$.

The best we can get is to find some bounds on the price of the Asian lookback option. Clearly

$$\begin{aligned}&\frac{1}{T}\int_0^T \max\left\{\left(P-\max_{u\leqslant s\leqslant T} X_s\right),0\right\}\,du\\ \leqslant\ &\max\left\{(P-X_T),0\right\}.\end{aligned}$$

This is just the payout of a European put. We can also estimate

$$\begin{aligned}&\frac{1}{T}\int_0^T \max\left\{\left(P-\max_{u\leqslant s\leqslant T} X_s\right),0\right\}\,du\\ \leqslant\ &\frac{1}{T}\int_0^T \max\left\{(P-X_u),0\right\}\,du\end{aligned}$$

which gives the payout of an Asian option. The other bound:

$$\begin{aligned}&\frac{1}{T}\int_0^T \max\left\{\left(P-\max_{u\leqslant s\leqslant T} X_s\right),0\right\}\,du\\ \geqslant\ &\frac{1}{T}\int_0^T \max\left\{\left(P-\max_{0\leqslant s\leqslant T} X_s\right),0\right\}\,du\\ =\ &\max\left\{\left(P-\max_{0\leqslant s\leqslant T} X_s\right),0\right\}.\end{aligned}$$

This is the payout of a lookback option. The value of such an option is well-known. Therefore we can say that the value of our option lies between the value of an Asian option and a lookback option.

3.3 Heavy-tailedness of the distribution of the relative limit order price

The distribution of the distance of arriving orders to the current stock price is heavy-tailed based on empirical observations, see e.g. [BMP02]. A natural question is therefore whether the distance of orders which are in the book is still heavy-tailed. We will investigate this question and consider for simplicity only sell orders which arrive at time 0. We fix a time horizon T and assume that one sell limit order arrives at time 0. The distribution of its distance to the current stock price is denoted by R_0, the corresponding density by f_{R_0}. Furthermore, the following assumptions are made:

Assumptions 3.12 *The stock price process $(X_t)_{t\in[0,T]}$ is independent of R_0, R_0 satisfies $P(R_0 \geqslant p) > 0$ for all $p \in \mathbb{R}$ and the following relations hold:*

1. *There exists some function $h:\mathbb{R}\to\mathbb{R}$ which satisfies $\lim_{p\to\infty} h(p) = \infty$ such that*

$$\left(P(R_0\geqslant p)-\int_p^\infty P\left(\sup_{0\leqslant s\leqslant T} X_s\leqslant y\right) f_{R_0}(y)\ dy\right)\cdot h(p)\to 0 \qquad \text{as} \qquad p\to\infty, \tag{3.1}$$

2.

$$\int_{X_0}^{\infty} P\left(\sup_{0\leqslant s\leqslant T} X_s \leqslant y\right) f_{R_0}(y)\ dy \neq 0$$

and

3.

$$\int_{p}^{\infty} P\left(\sup_{0\leqslant s\leqslant T} X_s \leqslant y\right) f_{R_0}(y)\ dy < 1, \qquad \text{for } all\ p \geqslant X_0.$$

Remark 3.13 *Those assumptions are e.g. satisfied if R_0 is independent of $(X_t)_{t\in[0,T]}$, R_0 has a power law distribution with density*

$$f_{R_0}(p) = \begin{cases} \left(\frac{a}{b+p}\right)^{\lambda} & for \quad p \geqslant X_0 \\ 0 & for \quad p \geqslant X_0 \end{cases}, \qquad b \in \mathbb{R},\ a, \lambda \in \mathbb{R}_+,$$

and the stock price is given by (2.2)*. In that case, we could choose $h(p) = p^n$ for some $n \in \mathbb{N}$.*

We always have that (3.1) holds for $h(p) = 0$ if X and R_0 are independent because

$$P(R_0 \geqslant p) \geqslant \int_{p}^{\infty} P\left(\sup_{0\leqslant s\leqslant T} X_s \leqslant y\right) f_{R_0}(y)\ dy$$

and

$$P(R_0 \geqslant p) \to 0 \qquad \text{as} \qquad p \to \infty.$$

Now we are interested in the heavy-tailedness of the distribution of R_0 conditional on the limit order not being executed until time T.

Theorem 3.14 *Under Assumptions* (3.12) *the distribution F of R_0 conditional on the limit order not being executed until time T is heavy-tailed.*

Proof. We compute

$$\begin{aligned} & P\left(R_0 \leqslant p \;\middle|\; \sup_{0\leqslant s\leqslant T} X_s \leqslant R_0\right) \\ = \; & \frac{\int_{X_0}^{p} P\left(\sup_{0\leqslant s\leqslant T} X_s \leqslant y\right) f_{R_0}(y)\ dy}{\int_{X_0}^{\infty} P\left(\sup_{0\leqslant s\leqslant T} X_s \leqslant y\right) f_{R_0}(y)\ dy}. \end{aligned}$$

Let $a := \int_{X_0}^{\infty} P\left(\sup_{0\leqslant s\leqslant T} X_s \leqslant y\right) f_{R_0}(y)\ dy$. Then

$$\begin{aligned} & \lim_{p\to\infty} \frac{\bar{F}(p+y)}{\bar{F}(p)} = \lim_{p\to\infty} \frac{1 - \frac{1}{a}\int_{p+y}^{\infty} P\left(\sup_{0\leqslant s\leqslant T} X_s \leqslant y\right) f_{R_0}(y)\ dy}{1 - \frac{1}{a}\int_{p}^{\infty} P\left(\sup_{0\leqslant s\leqslant T} X_s \leqslant y\right) f_{R_0}(y)\ dy} \\ \geqslant \; & \lim_{p\to\infty} \frac{1 - \frac{1}{a}P(R_0 \geqslant p+y)}{1 - \frac{1}{a}P(R_0 \geqslant p) + \frac{1}{a}\frac{\epsilon}{h(p)}} = \lim_{p\to\infty} \frac{a - P(R_0 \geqslant p+y)}{a - P(R_0 \geqslant p) + \frac{\epsilon}{h(p)}} \\ = \; & \lim_{p\to\infty} \left(\frac{a}{a - P(R_0 \geqslant p) + \frac{\epsilon}{h(p)}} - \frac{P(R_0 \geqslant p+y)}{a - P(R_0 \geqslant p) + \frac{\epsilon}{h(p)}} \right) = 1. \end{aligned}$$

The other inequality follows trivially and therefore the lemma is shown. ■

Remark 3.15 *If we include cancellation, we get the same result, provided we assume that cancellation is independent of* R_0 *and* $(X_t)_{t\in[0,T]}$. *Clearly, if we look at limit orders which are submitted at an arbitrary point in time, we can proceed similarly.*

3.4 Cancellation and queuing theory

In a simplified setting, we have a more detailed look at the issue of cancellation. As usual, only sell orders are considered since buy orders are treated similarly. We use that cancellation in the limit order book can be treated as an $M/G/\infty/\infty$ queuing model, using Kendall's notation. This means that all the arriving customers (limit orders) can be served (stored in the book) immediately because the number of service channels is infinite. G is the probability distribution which denotes the time to cancellation. It is the same for all orders. We only look at how we have to modify the arrival sell order book.

Lemma 3.16 *Assume that orders arrive according to a Poisson process at a rate of* λ *and are cancelled according to the distribution* G. *Then the probability that there are* k *orders submitted and not yet cancelled until time* t *is given by*

$$P\left(k \text{ orders in the arrival book at time } t\right) = \frac{(p\lambda t)^k}{k!}e^{-p\lambda t}$$

where

$$p\lambda t = \lambda \int_0^t \left(1 - G\left(s\right)\right)\, ds.$$

Remark 3.17 *In particular, if we assume an exponential cancellation time distribution with parameter* $\frac{1}{\mu}$,

$$G\left(t\right) = 1 - e^{-\frac{t}{\mu}},$$

we have

$$p\lambda t = \lambda\mu\left(1 - e^{-\frac{t}{\mu}}\right).$$

Proof. Assume that there are n orders which have arrived during the time interval $[0,t]$. The probability that an order has not been cancelled until time t (Recall that we are looking at the arrival book and therefore have no cancellation.) is

$$\begin{aligned} p &= \int_0^t \left(1 - G\left(t-s\right)\right)\, \frac{ds}{t} \\ &= \frac{1}{t}\int_0^t \left(1 - G\left(s\right)\right)\, ds. \end{aligned}$$

Then the probability that there are k orders being in the arrival book at time t is

$$\begin{aligned} & P\left(k \text{ orders in the arrival book at time } t\right) \\ = & \sum_{n=k}^{\infty} \binom{n}{k} p^k \left(1-p\right)^{n-k} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\ = & \frac{(p\lambda t)^k}{k!}e^{-p\lambda t}, \qquad k = 0,1,2,\ldots \end{aligned}$$

i.e. the probability that k orders are in the book at time t follows a Poisson distribution with mean

$$p\lambda t = \lambda \int_0^t (1 - G(x))\ dx.$$

■

In other words, the above lemma tells us that, instead of looking at the arrival process being a Poisson process with arrival rate λ, we use a Poisson process with arrival rate $\lambda \int_0^t (1 - G(s))\ ds$. From there on, we proceed as before.

Clearly, we could also take into account different cancellation rates for orders with different distances to the current stock price as well as non-homogeneous Poisson arrival processes. If we assume that the arrival intensity is given by $\lambda_t := \alpha_\theta^s ([0,t], [0,\infty), [0,\infty))$, we obtain

$$p = \frac{1}{\lambda_t} \int_0^t (1 - G(x))\ d\lambda_x$$

and

$$P(k \text{ orders in the arrival book at time } t) = \frac{(p\lambda_t)^k}{k!} e^{-p\lambda_t}, \qquad k = 0, 1, 2, \ldots$$

Remark 3.18 *An empirical analysis of cancellation rates can be found in Potters and Bouchaud [PB03b]. The most important result is the dependence of the cancellation rate on the distance from the current bid (or ask). The life-time of a given order increases as one moves away from the bid (ask). They also provide a reasonable intuitive explanation: Far away orders are typically put in the market by patient investors that want to take advantage of stock price changes in the medium to long-term. Orders at and around the bid and ask prices correspond to very active market participants that observe the market price permanently and readjust their order at a very high frequency.*

We finish this subsection by looking at the limiting distribution of the number of arriving orders which are not cancelled.

Lemma 3.19 *Assume that G has a finite mean denoted by μ. Then*

$$\begin{aligned} & \lim_{t\to\infty} P(k \textit{ orders in the arrival book at time } t) \\ = \ & e^{-\lambda\mu} \frac{(\lambda\mu)^k}{\mu!}, \qquad k = 0, 1, 2, \ldots \end{aligned}$$

The amazing result here is that the limiting distribution does not require the shape of the distribution of G, but only its mean.

Proof. For the proof see also [Tij03]. Assume that there are no orders cancelled at time 0 and define for any $t > 0$:

$$p_j(t) = P(j \text{ orders in the book at time } t), \qquad j = 0, 1, 2, \ldots$$

Consider now $p_j(t + \Delta t)$ for Δt small. The event that there are j orders in the book at time $t + \Delta t$ can occur in the following four mutually exclusive ways:

1. no arrival occurs in $(0, \Delta t)$ and there are j orders in the book at time $t + \Delta t$ due to arrivals in $(\Delta t, t + \Delta t)$,

2. one arrival occurs in $(0, \Delta t)$, the first order is cancelled before time $t + \Delta t$ and there are

j orders in the book at time $t + \Delta t$ due to arrivals in $(\Delta t, t + \Delta t)$,
3. one arrival occurs in $(0, \Delta t)$, the first arrival is not cancelled before time $t + \Delta t$ and there are $j - 1$ other orders at time $t + \Delta t$ due to arrivals in $(\Delta t, t + \Delta t)$,
4. two or more arrivals occur in $(0, \Delta t)$ and j orders are in the book at time $t + \Delta t$.
Let $B(t)$ denote the probability distribution of the cancellation time of an order. Then, since a probability distribution function has at most a countable number of discontinuity points, we find for almost all $t > 0$ that

$$\begin{aligned} p_j(t + \Delta t) &= (1 - \lambda \Delta t) p_j(t) + \lambda \Delta t B(t + \Delta t) p_j(t) \\ &\quad + \lambda \Delta t [1 - B(t + \Delta t)] p_{j-1}(t) + o(\Delta t). \end{aligned}$$

Subtracting $p_j(t)$ from $p_j(t + \Delta t)$, dividing by Δt and letting $\Delta t \to 0$, we find

$$\begin{aligned} p_0'(t) &= -\lambda (1 - B(t)) p_0(t) \\ p_j'(t) &= -\lambda (1 - B(t)) p_j(t) + \lambda (1 - B(t)) p_{j-1}(t), \qquad j = 1, 2, \ldots \end{aligned}$$

Next, by induction on j, it is readily verified that

$$p_j(t) = e^{-\lambda \int_0^t 1 - B(x)\ dx} \frac{\left[\lambda \int_0^t 1 - B(x)\ dx\right]^j}{j!}, \qquad j = 0, 1, \ldots$$

By a continuity argument this relation holds for all $t \geqslant 0$. Since $\int_0^\infty 1 - B(x)\ dx = \mu$, the result follows. ■

3.5 The waiting time until execution or cancellation

In this subsection, we want to answer questions such as: How long do we have to wait until execution? How long is this compared to the average waiting time until cancellation? What is the chance that the order is actually executed, if at all, before it is deleted. We only look at sell orders, buy orders are again treated similarly.

Definition 3.20 (Time to execution and time to cancellation) *We define the* ***time to execution*** *of an order, submitted at time* 0, *with limit price* p *as*

$$T^p := \inf\{t \geqslant 0 \mid X_t \geqslant p\}.$$

The ***time to cancellation*** *is denoted by* τ^p, *i.e. the order is cancelled at the (random) time* τ^p.

Assumptions 3.21 *1. For simplicity and to obtain explicit results, we consider the following stock price process:* $(X_t)_{t \geqslant 0}$ *is given by*

$$X_t = X_0 \exp\left\{\sigma^2 \nu t + \sigma W_t\right\}$$

with $\sigma \in \mathbb{R}_+$, $\nu \in \mathbb{R}$ *and* $(W_t)_{t \geqslant 0}$ *a standard Brownian motion.*

2. For the time to cancellation, we assume that τ^p *has an exponential distribution with parameter* λ_p, *i.e.*

$$P(\tau^p \leqslant t) = 1 - e^{-\lambda_p t}, \qquad t \geqslant 0.$$

3. *To avoid lengthy special cases, which are straightforward, assume that* $P(T^p < \infty) = 1$, *which is equivalent to* $\nu \geqslant 0$.

4. *Assume that* τ^p *and* T^p *are independent.*

5. *We only look at sell orders here, therefore* $p \geqslant X_0$ *and investigate the behaviour of one limit order which is submitted at time* 0 *with price* $p \in \mathbb{R}_+$.

We compute several properties related to T^p and τ^p. For this, we use the following terms:

- **Time to execution** T^p: The time, when the order is executed, not considering cancellation.
- **Time to cancellation** τ^p: The time, when the order is cancelled, not considering execution.
- **Time to deletion** $\min(T^p, \tau^p)$: the time, when the order is either cancelled or executed.

The following well-known result is used, see also [BS02]:

Lemma 3.22 *Under Assumptions* (3.21) *the density of the time to execution is given by*

$$\begin{aligned} & P(T^p \in dt) \\ = & \frac{\ln\left(\frac{p}{X_0}\right)}{\sigma\sqrt{2\pi}t^{\frac{3}{2}}}\left(\frac{p}{X_0}\right)^{\nu}\exp\left(-\frac{\nu^2\sigma^2 t}{2}-\frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 t}\right)dt \end{aligned}$$

for $p \geqslant X_0$.

We also want to know when the order is deleted from the book, either by execution or cancellation.

Lemma 3.23 *Under Assumptions* (3.21) *the time to deletion is given by* $\min\{\tau^p, T^p\}$. *Then*

$$\begin{aligned} & P(\min\{\tau^p, T^p\} \leqslant t) \\ = & 1 - e^{-\lambda_p t}\int_t^{\infty}\frac{\ln\left(\frac{p}{X_0}\right)}{\sigma\sqrt{2\pi}t^{\frac{3}{2}}}\left(\frac{p}{X_0}\right)^{\nu}\exp\left(-\frac{\nu^2\sigma^2 t}{2}-\frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 t}\right)dt. \end{aligned}$$

Proof.

$$\begin{aligned} & P(\min\{\tau^p, T^p\} \leqslant t) \\ = & 1 - P(\tau^p \geqslant t)\,P(T^p \geqslant t) \\ = & 1 - e^{-\lambda_p t}\int_t^{\infty}\frac{\ln\left(\frac{p}{X_0}\right)}{\sigma\sqrt{2\pi}t^{\frac{3}{2}}}\left(\frac{p}{X_0}\right)^{\nu}\exp\left(-\frac{\nu^2\sigma^2 t}{2}-\frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 t}\right)dt. \end{aligned}$$

■

Next, we state the chance of being executed (and not cancelled earlier):

Lemma 3.24 *Under Assumptions* (3.21) *an order with price p has a probability of being executed, given by*

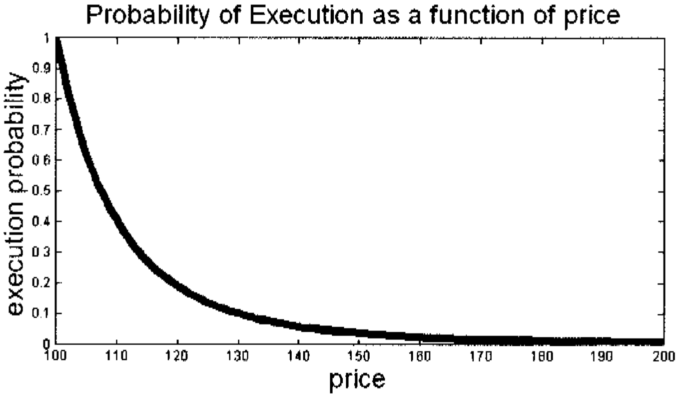
$$P\left(\mathcal{T}^p \leqslant \tau^p\right) = \left(\frac{X_0}{p}\right)^{\sqrt{\nu^2 + \frac{2\lambda_p}{\sigma^2}} - \nu}.$$

Proof. Observe that

$$P\left(\sup_{0 \leqslant s \leqslant \tau^p} X_s \geqslant p\right) = P\left(\mathcal{T}^p \leqslant \tau^p\right).$$

The rest follows from well-known results about $P\left(\sup_{0\leqslant s\leqslant \tau^p} X_s \geqslant p\right)$. ■

With the parameters $\mu = 0.08$, $\sigma = 0.2$, $X_0 = 100$, $\lambda_p = \frac{200}{p}$, we obtain the following graph:



The next remark gives us a nice condition on the parameter λ_p which would lead to such impatient traders that the probability of execution decreases indeed if we approach X_0.

Remark 3.25 *Assume that λ_p satisfies*

$$-\sqrt{\nu^2 + \frac{2\lambda_p}{\sigma^2}} + \nu + p \ln\left(\frac{X_0}{p}\right)\left(\frac{1}{\sqrt{\nu^2 + \frac{2\lambda_p}{\sigma^2}}}\frac{1}{\sigma^2}\frac{d}{dp}\lambda_p\right) > 0.$$

Then $\left(\frac{X_0}{p}\right)^{\sqrt{\nu^2+\frac{2\lambda_p}{\sigma^2}}-\nu}$ *is strictly increasing as a function of p. This shows that orders which are closer to X_0 have a higher chance of being cancelled before they are actually executed. This corresponds to the fact that traders which submit orders further away from the current stock price are less impatient. If e.g. $\lambda_p \leqslant c$ for some $c \in \mathbb{R}$ and all $p \geqslant X_0$, we get:*

$$\lim_{p\to\infty} P\left(\tau^p \geqslant \mathcal{T}^p\right) = 0.$$

The chance of being cancelled and not executed earlier is:

Lemma 3.26 *Under Assumptions* (3.21) *the probability of an order with price p of being cancelled is given by*

$$P\left(\mathcal{T}^p \geqslant \tau^p\right) = 1 - \left(\frac{X_0}{p}\right)^{\sqrt{\nu^2 + \frac{2\lambda_p}{\sigma^2}} - \nu}.$$

If we fix a finite time horizon T, we can compute several quantities related to τ^p and T^p. To illustrate this, we pick one example. One of our interests lies in the probability that our order is actually executed before time T (and not deleted before execution). Therefore we finish this subsection with the following lemma.

Lemma 3.27 *Under Assumptions* (3.21),

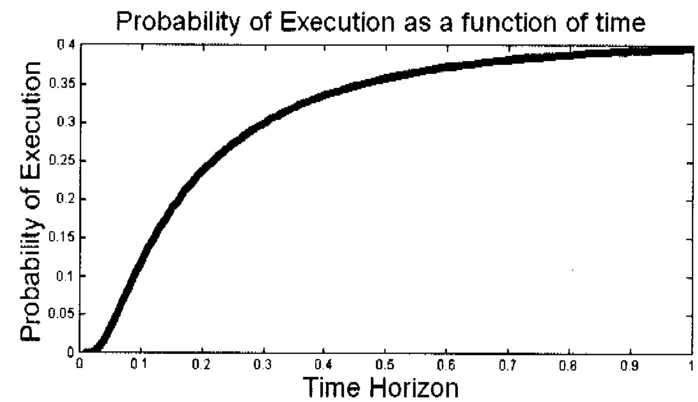
$$P(\tau^p \geqslant T^p \text{ and } T^p \leqslant T) = \frac{\ln\left(\frac{p}{X_0}\right)}{\sigma\sqrt{2\pi}}\left(\frac{p}{X_0}\right)^{\nu}\int_0^T \frac{e^{-\lambda_p t}}{t^{\frac{3}{2}}}\exp\left(-\frac{\nu^2\sigma^2 t}{2}-\frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 t}\right)dt.$$

Proof.

$$\begin{aligned}
&P(\tau^p \geqslant T^p \text{ and } T^p \leqslant T)\\
&= \int_0^T P(\tau^p \geqslant T^p \mid T^p = t)\, P(T^p = t)\ dt = \int_0^T P(\tau^p \geqslant t)\, P(T^p = t)\ dt\\
&= \int_0^T e^{-\lambda_p t}\frac{\ln\left(\frac{p}{X_0}\right)}{\sigma\sqrt{2\pi}t^{\frac{3}{2}}}\left(\frac{p}{X_0}\right)^{\nu}\exp\left(-\frac{\nu^2\sigma^2 t}{2}-\frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 t}\right)dt\\
&= \frac{\ln\left(\frac{p}{X_0}\right)}{\sigma\sqrt{2\pi}}\left(\frac{p}{X_0}\right)^{\nu}\int_0^T \frac{e^{-\lambda_p t}}{t^{\frac{3}{2}}}\exp\left(-\frac{\nu^2\sigma^2 t}{2}-\frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 t}\right)dt.
\end{aligned}$$

■

With the parameters $X_0 = 100$, $p = 105$, $\lambda_p = \frac{200}{p}$, $\mu = 0.08$, $\sigma = 0.2$, we obtain (Note that a large percentage of orders is cancelled before they can be executed.)



3.6 Applications to the volume-volatility relation

We want to investigate the relation between the volume of orders in the book and the volatility of the stock price. It turns out that it is not so much the volatility but the distribution of the maximum of the stock price which influences the volume of orders. This would be an aspect which could be investigated empirically. For simplicity, we do not consider the size of the orders here and use therefore

$$f_{(T,R)}(u,v) = \int_0^\infty f_{(T,R,S)}(u,v,w)\ dw.$$

As calculated in (2.23), we get:

$$E\left(\mathcal{O}_\theta^s(\{t\},[0,p],[0,\infty),[0,\infty))\right) = m_\theta^s \cdot \alpha_\theta^s(E)$$

where

$$m_\theta^s = \int_0^t \int_{\max_{u\leqslant s\leqslant t} x_s - x_u}^{p-x_u} f_{(T,R)}(u,v)\ dv\ du.$$

Then the variance of the intensity, which is a measure for the variance of the volume of orders in the book, is given by

$$Var\left(\alpha^s(E)\cdot\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right)$$

Therefore, we see the influence of the term $\max_{u\leqslant s\leqslant t} X_s - X_u$.

Empirical Conjecture We conjecture that, given p, the characteristics

$$E\left(X_u\chi_{\{X_u\leqslant p\}}\right)$$
$$P\left(X_u \leqslant p\right)$$

and

$$E\left(\left(\max_{u\leqslant s\leqslant t} X_s - X_u\right)\chi_{\left\{\max_{u\leqslant s\leqslant t} X_s\leqslant p\right\}}\right)$$

influence the mean number of orders in the order book. This would have to be tested empirically. (The smaller $E\left(X_u\chi_{\{X_u\leqslant p\}}\right)$ and $E\left(\left(\max_{u\leqslant s\leqslant t} X_s - X_u\right)\chi_{\left\{\max_{u\leqslant s\leqslant t} X_s\leqslant p\right\}}\right)$ and the larger $P\left(X_u\leqslant p\right)$, the larger is the mean number of orders.).

Proof. Fix $p \in \mathbb{R}_+$. Clearly, the absolute size plays a role, therefore assume that we have two stock price processes $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ with $E(X_t) = E(Y_t)$ for all $t \geqslant 0$. Also assume that

$$f_{(T,R)}(u,v) = \frac{1}{T\cdot P}\chi_{\{u\leqslant T\}}\chi_{\{v\leqslant P\}}$$

and that $\alpha^s(E)$ is nonrandom. Then the intensity is given by

$$\begin{aligned}
& E\left(m_\theta^s\cdot\alpha^s(E)\right) = \alpha^s(E)\cdot E\left(m_\theta^s\right)\\
= \ & \alpha^s(E)\cdot E\left(\left(\int_0^t\int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u}\frac{1}{T\cdot P}\chi_{\{u\leqslant T\}}\chi_{\{v\leqslant P\}}\ dw\ dv\ du\right)\right)\\
= \ & \alpha^s(E)\cdot E\left(\frac{1}{T\cdot P}\left(\int_0^{t\wedge T}\left((p-X_u)\wedge P\right)\cdot\chi_{\{X_u\leqslant p\}}\ du\right)\right)\\
& -\alpha^s(E)\cdot E\left(\frac{1}{T\cdot P}\left(\int_0^{t\wedge T}\left(\max_{u\leqslant s\leqslant t} X_s - X_u\right)\chi_{\left\{\max_{u\leqslant s\leqslant t} X_s - X_u\leqslant p - X_u\right\}}\ du\right)\right).
\end{aligned}$$

Assume that $t \leqslant T$ and $p \leqslant P$. Then

$$\begin{aligned}
& E\left(m_\theta^s \cdot \alpha^s(E)\right) \\
= \;& \alpha^s(E) \cdot \frac{1}{T \cdot P} \times \int_0^t E\left((p - X_u)\right) \cdot \chi_{\{X_u \leqslant p\}}\, du \\
& - \alpha^s(E) \cdot \frac{1}{T \cdot P} \times \int_0^t E\left(\max_{u \leqslant s \leqslant t} X_s - X_u\right) \chi_{\left\{\max_{u \leqslant s \leqslant t} X_s - X_u \leqslant p - X_u\right\}}\, du \\
= \;& \alpha^s(E) \cdot \frac{1}{T \cdot P} \int_0^t p \cdot P\left(X_u \leqslant p\right) - E\left(X_u \cdot \chi_{\{X_u \leqslant p\}}\right)\, du \\
& - \alpha^s(E) \cdot \frac{1}{T \cdot P} \int_0^t E\left(\left(\max_{u \leqslant s \leqslant t} X_s - X_u\right) \chi_{\left\{\max_{u \leqslant s \leqslant t} X_s - X_u \leqslant p - X_u\right\}}\right)\, du.
\end{aligned}$$

We compare this to the average intensity if we have another stock price process $(Y_t)_{t \geqslant 0}$. Assume that the processes $(X_t)_{t \geqslant 0}$ and $(Y_t)_{t \geqslant 0}$ are such that for all $0 \leqslant u \leqslant t$:

$$\begin{aligned}
E\left(X_u \chi_{\{X_u \leqslant p\}}\right) & \leqslant E\left(Y_u \chi_{\{Y_u \leqslant p\}}\right), \\
P\left(X_u \leqslant p\right) & \geqslant P\left(Y_u \leqslant p\right)
\end{aligned}$$

and

$$E\left(\left(\max_{u \leqslant s \leqslant t} X_s - X_u\right) \chi_{\left\{\max_{u \leqslant s \leqslant t} X_s \leqslant p\right\}}\right) \leqslant E\left(\left(\max_{u \leqslant s \leqslant t} Y_s - Y_u\right) \chi_{\left\{\max_{u \leqslant s \leqslant t} Y_s \leqslant p\right\}}\right).$$

Then

$$\begin{aligned}
& E\left(m_\theta(t, p, \infty, s) \cdot \alpha^s(E)\right) \\
= \;& \alpha^s(E) \cdot \frac{1}{T \cdot P} \int_0^t p \cdot P\left(Y_u \leqslant p\right) - E\left(Y_u \cdot \chi_{\{Y_u \leqslant p\}}\right)\, du \\
& - \alpha^s(E) \cdot \frac{1}{T \cdot P} \int_0^t E\left(\left(\max_{u \leqslant s \leqslant t} Y_s - Y_u\right) \chi_{\left\{\max_{u \leqslant s \leqslant t} Y_s \leqslant p\right\}}\right)\, du
\end{aligned}$$

and

$$\begin{aligned}
& \alpha^s(E) \cdot \frac{1}{T \cdot P} \int_0^t p \cdot P\left(Y_u \leqslant p\right) - E\left(Y_u \cdot \chi_{\{Y_u \leqslant p\}}\right)\, du \\
& - \alpha^s(E) \cdot \frac{1}{T \cdot P} \int_0^t E\left(\left(\max_{u \leqslant s \leqslant t} Y_s - Y_u\right) \chi_{\left\{\max_{u \leqslant s \leqslant t} Y_s \leqslant p\right\}}\right)\, du \\
\leqslant \;& \alpha^s(E) \cdot \frac{1}{T \cdot P} \cdot \int_0^t p \cdot P\left(X_u \leqslant p\right) - E\left(X_u \cdot \chi_{\{X_u \leqslant p\}}\right)\, du \\
& - \alpha^s(E) \cdot \frac{1}{T \cdot P} \cdot \int_0^t E\left(\left(\max_{u \leqslant s \leqslant t} X_s - X_u\right) \chi_{\left\{\max_{u \leqslant s \leqslant t} X_s \leqslant p\right\}}\right)\, du.
\end{aligned}$$

■

We show that the number of sell orders in the book with absolute price less than p ($p > 0$ arbitrary) converges to zero as the time horizon goes to infinity.

Lemma 3.28 *Assume that* $(X_t)_{t\geqslant 0}$ *is such that*

$$P\left(\max_{s\geqslant u} X_s \geqslant p\right) = 1, \qquad \text{for } all\ u \geqslant 0$$

and that there exists $\bar{T} > 0$ *such that* $f_{(T,R)}(u,v) = 0$ *for all* $u \geqslant \bar{T}$. *Then*

$$\lim_{t\to\infty} E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) = 0$$

and

$$\lim_{t\to\infty} Var\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) = 0.$$

Proof. Fix $\epsilon > 0$. Let $t \geqslant \bar{T}$ and assume that t is so large that

$$P\left(\max_{u\leqslant s\leqslant t} X_s \geqslant p\right) \geqslant 1-\epsilon$$

for all $u \leqslant \bar{T}$. Such a t exists by assumption. Then

$$\begin{aligned}
& E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) \\
= \ & E\left(\int_0^{\bar{T}} \int_{\max_{u\leqslant s\leqslant t} X_s}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) \\
\leqslant \ & \epsilon E\left(\int_0^{\bar{T}} \int_0^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) \leqslant \epsilon
\end{aligned}$$

Therefore,

$$\lim_{t\to\infty} E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) = 0$$

Furthermore,

$$\begin{aligned}
& E\left(\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right)^2\right) \\
\leqslant \ & \epsilon E\left(\left(\int_0^t \int_0^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right)^2\right) \leqslant \epsilon.
\end{aligned}$$

Therefore, also

$$\lim_{t\to\infty} Var\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) = 0.$$

■

Remark 3.29 *Clearly, we do not need the assumption that there exists* $\bar{T} \geqslant 0$ *such that* $f_{(T,R)}(u,v) = 0$ *for all* $u \geqslant \bar{T}$. *Instead we impose that for every* $\epsilon > 0$ *there exists* $\bar{T} \geqslant 0$ *such that*

$$\int_{\bar{T}}^{\infty} \int_{0}^{\infty} f_{(T,R)}(u,v)\ dv\ du \leqslant \epsilon$$

which is clearly satisfied, since f *is integrable.*

Proof. The proof uses similar techniques as the previous one. ■

Remark 3.30 *We have shown that* $\mathcal{O}^s_\theta(\{t\} \times [0,p] \times [0,\infty) \times [0,s])$ *has a Poisson distribution with mean* $\alpha^s_\theta \cdot m_\theta(t,p,\infty,s)_\theta$. *Therefore, the variance of* $\mathcal{O}^s_\theta(\{t\} \times [0,p] \times [0,\infty) \times [0,s])$ *is also given by* $m_\theta(t,p,\infty,s)_\theta \cdot \alpha^s_\theta$. *From this we see that the variance of the order book also depends in a non-trivial way on the behaviour of the maximum of the stock price.*

3.7 Maximum of the order-book in the long run

Empirically, one has found, that the order book has a maximum "a bit away" from the current stock price, see e.g. [BMP02]. Here we want to analytically find the quantity which is necessary to check these findings. If the distribution of the arrival order distance is according to a power law, the maximum of the arriving orders is at the current stock price. However, these orders also have a larger probability of being executed. Therefore it is a priori not clear which one is the resulting distribution. The distribution of the relative price of incoming orders at time t is denoted by R_t. We assume that the density of R_t is given by

$$f_t(p) = \begin{cases} \frac{a}{(b+p)^\lambda} & \text{for} \quad a,b \geqslant 0,\ X_t \leqslant p \leqslant p_{\max} \\ 0 & \text{for} \quad p \geqslant p_{\max} \end{cases},$$

where $p_{\max} = \inf\left\{p \,\middle|\, \int_{X_t}^{p} f_t(p)\ dp = 1\right\}$. Empirically it was found that $\lambda \approx 1.6$ (see [BMP02]).

To get analytical solutions we assume as simplification that R_t is independent of time and the stock price process X_t. What is the long-term distribution of D_t, where D_t denotes the distribution of the distance of orders which are in the order book (after cancellation and execution)? Assume D_t has the density f_{D_t}. For $d \geqslant 0$, this means

$$P(D_t \leqslant d) = \int_0^d f_{D_t}(u)\ du.$$

We observe the following: If an order exists in the order book at time t and it was submitted at time s ($s \leqslant t$), then in addition, it should not have been cancelled or executed in the mean time. Here we consider the long-term (average) behaviour. Therefore, we only look at the average order book (averaged over time), in other words, we assume that orders arrive with a constant rate of 1 per unit time (the precise number of course varies from stock to stock and is not relevant for our analytical purposes here) and are cancelled (we need cancellation here, otherwise our order book usually blows up in the long run) at a rate of c per unit time. Overall, we assume that one order submitted at time s survives until time t with probability $\exp(-c(t-s))$. Denote by $f_{\tilde{D}_t}(u,s)$ the joint density of time and distance (order arrives at time s and has distance u at time t, provided the order still exists at time t). $f_{X_t}(v \mid X_{s'} - X_t \leqslant u \ \forall s' \in [s,t])$ denotes the conditional density that $X_s = X_t - v$, provided the stock price has not exceeded $X_t + u$ in the mean time. Then the order was submitted at a distance $u+v$ at time s (with density $f_s(u+v)$).

$$\begin{aligned} P(D_t \leqslant d) &= \int_0^d f_{D_t}(u)\ du \\ &= \int_0^d \int_0^t \exp(-c(t-s)) f_{\tilde{D}_t}(u,s)\ ds\ du \\ &= \int_0^d \int_0^t \exp(-c(t-s)) \int_{-u}^{\infty} f_s(u+v) f_{X_t}(v \,|X_{s'} - X_t \leqslant u \ \forall s' \in [s,t])\ dv\ ds\ du \end{aligned}$$

This can be calculated explicitly:

Theorem 3.31 *Assume that the stock price process $(X_t)_{t\geqslant 0}$ is given by (2.2). Then*

$$\begin{aligned} & P(D_t \leqslant d) \\ = & \int_0^d \int_0^T \int_0^\infty \int_0^{u+z} \exp(-c(t-s)) P\left(X_T \in dz,\ X_t \in da \ \Big|\ \sup_{t \leqslant s \leqslant T} X_s \leqslant u + z \right) \\ & \times\ f_t(u+z-a)\ da\ dz\ dt\ du \end{aligned}$$

where

$$P\left(X_T \in dz,\ X_t \in da \ \Big|\ \sup_{t \leqslant s \leqslant T} X_s \leqslant u + z \right)$$

is given by

$$\begin{aligned} & P\left(X_T \in dz,\ X_t \in da \ \Big|\ \sup_{t \leqslant s \leqslant T} X_s \leqslant u + z \right) \\ = & \left(\frac{1}{z\sigma\sqrt{2\pi(T-t)}} e^{-\frac{\left(\ln\left(\frac{z}{a}\right) + \frac{\sigma^2}{2}(T-t)\right)^2}{2\sigma^2(T-t)}}\ dz - \frac{\sqrt{a}}{z^{\frac{3}{2}}\sigma\sqrt{2\pi(T-t)}} e^{-\frac{\sigma^2(T-t)}{8}} e^{-\frac{\left(\ln\frac{(u+z)^2}{za}\right)^2}{2\sigma^2(T-t)}}\ dz \right) \\ \times & \ \frac{1}{a\sigma\sqrt{2\pi t}} e^{-\frac{\left(\ln\left(\frac{a}{X_0}\right) + \frac{\sigma^2}{2}t\right)^2}{2\sigma^2 t}}\ da \cdot \frac{1}{\mathcal{N}\left(\frac{\ln\left(\frac{u+z}{X_0}\right) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) - \frac{X_0}{u+z}\mathcal{N}\left(\frac{-\ln\left(\frac{u+z}{X_0}\right) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)}. \end{aligned}$$

Proof. The proof is described in the remaining part of this subsection. ■

Theorem (3.31) is shown after using multiplication by $f_t(u+z-a)$ and integration and the following lemma:

Lemma 3.32 *Assume that the stock price process $(X_t)_{t\geqslant 0}$ is given by (2.2). We have the* *following result:*

$$P\left(X_T \in dz,\ X_t \in da \ \middle|\ \sup_{t\leqslant s\leqslant T} X_s \leqslant u+z\right)$$
$$= \left(\frac{1}{z\sigma\sqrt{2\pi\,(T-t)}}e^{-\frac{\left(\ln\left(\frac{z}{a}\right)+\frac{\sigma^2}{2}(T-t)\right)^2}{2\sigma^2(T-t)}}\,dz - \frac{\sqrt{a}}{z^{\frac{3}{2}}\sigma\sqrt{2\pi\,(T-t)}}e^{-\frac{\sigma^2(T-t)}{8}}e^{-\frac{\left(\ln\frac{(u+z)^2}{za}\right)^2}{2\sigma^2(T-t)}}\,dz\right)$$
$$\times\ \frac{1}{a\sigma\sqrt{2\pi t}}e^{-\frac{\left(\ln\left(\frac{a}{X_0}\right)+\frac{\sigma^2}{2}t\right)^2}{2\sigma^2 t}}\,da\cdot\frac{1}{\mathcal{N}\left(\frac{\ln\left(\frac{u+z}{X_0}\right)+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)-\frac{X_0}{u+z}\mathcal{N}\left(\frac{-\ln\left(\frac{u+z}{X_0}\right)+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)}.$$

The remainder of this section will deal with the proof of this lemma. In all of the following lemmas, we assume that the stock price process $(X_t)_{t\geqslant 0}$ is given by (2.2). If we omit the proof, the result is either straightforward or can be found in [BS02]. From elementary observations, we obtain the following results.

Lemma 3.33 *Let*

$$X_s = X_0 e^{\sigma^2\nu s+\sigma W_s}$$

with $\nu\in\mathbb{R}$. *Then*

$$P_x\left(X_t\in dz\right) = \frac{1}{z\sigma\sqrt{2\pi t}}e^{-\frac{\left(\ln\left(\frac{z}{x}\right)-\nu\sigma^2 t\right)^2}{2\sigma^2 t}}\,dz,$$

$$P_x\left(\sup_{0\leqslant s\leqslant t} X_s\geqslant y,\ X_t\in dz\right)$$
$$=\begin{cases}\frac{z^{\nu-1}}{x^{\nu}\sigma\sqrt{2\pi t}}e^{-\frac{\nu^2\sigma^2 t}{2}}e^{-\frac{\left(\ln\frac{z}{x}\right)^2}{2\sigma^2 t}}\,dz & y\leqslant z\\ \frac{z^{\nu-1}}{x^{\nu}\sigma\sqrt{2\pi t}}e^{-\frac{\nu^2\sigma^2 t}{2}}e^{-\frac{\left(\ln\frac{y^2}{zx}\right)^2}{2\sigma^2 t}}\,dz & z\leqslant y\end{cases}$$

and

$$P_x\left(\sup_{0\leqslant s\leqslant t}X_s\geqslant y\right)=\frac{1}{2}\mathrm{Erfc}\left(\frac{\ln\left(\frac{y}{x}\right)}{\sigma\sqrt{2t}}-\frac{\nu\sigma\sqrt{t}}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{y}{x}\right)^{2\nu}\mathrm{Erfc}\left(\frac{\ln\left(\frac{y}{x}\right)}{\sigma\sqrt{2t}}+\frac{\nu\sigma\sqrt{t}}{\sqrt{2}}\right)$$

where $\mathrm{Erfc}\,(x)=\frac{2}{\sqrt{\pi}}\int_x^{\infty}e^{-v^2}\,d\nu$. *With* $\nu=-\frac{1}{2}$, *we obtain:*

$$P_x\left(\sup_{0\leqslant s\leqslant t}X_s\leqslant y,\ X_t\in dz\right)$$
$$=\frac{1}{z\sigma\sqrt{2\pi t}}e^{-\frac{\left(\ln\left(\frac{z}{x}\right)+\frac{\sigma^2}{2}t\right)^2}{2\sigma^2 t}}\,dz-\frac{z^{\frac{-3}{2}}}{x^{-\frac{1}{2}}\sigma\sqrt{2\pi t}}e^{-\frac{\sigma^2 t}{8}}e^{-\frac{\left(\ln\frac{y^2}{zx}\right)^2}{2\sigma^2 t}}\,dz$$
$$=\frac{1}{z\sigma\sqrt{2\pi t}}e^{-\frac{\left(\ln\left(\frac{z}{x}\right)+\frac{\sigma^2}{2}t\right)^2}{2\sigma^2 t}}\,dz-\frac{\sqrt{x}}{z^{\frac{3}{2}}\sigma\sqrt{2\pi t}}e^{-\frac{\sigma^2 t}{8}}e^{-\frac{\left(\ln\frac{y^2}{zx}\right)^2}{2\sigma^2 t}}\,dz.$$

Based on those formulas, we obtain the next lemma.

Lemma 3.34 *We consider the case where $z \leqslant y$. Then*

$$P_x\left(\sup_{0\leqslant s\leqslant t} X_s \leqslant y,\ X_t \in dz\right) = \frac{1}{z\sigma\sqrt{2\pi t}}e^{-\frac{\left(\ln\left(\frac{z}{x}\right)-\nu\sigma^2 t\right)^2}{2\sigma^2 t}}\,dz - \frac{z^{\nu-1}}{x^{\nu}\sigma\sqrt{2\pi t}}e^{-\frac{\nu^2\sigma^2 t}{2}}e^{-\frac{\left(\ln\frac{y^2}{zx}\right)^2}{2\sigma^2 t}}\,dz$$

and

$$P_x\left(\sup_{t\leqslant s\leqslant T} X_s \leqslant y,\ X_T \in dz\,|X_t = a\right) = \frac{1}{z\sigma\sqrt{2\pi(T-t)}}e^{-\frac{\left(\ln\left(\frac{z}{a}\right)+\frac{\sigma^2}{2}(T-t)\right)^2}{2\sigma^2(T-t)}}\,dz - \frac{\sqrt{a}}{z^{\frac{3}{2}}\sigma\sqrt{2\pi(T-t)}}e^{-\frac{\sigma^2(T-t)}{8}}e^{-\frac{\left(\ln\frac{y^2}{za}\right)^2}{2\sigma^2(T-t)}}\,dz.$$

Proof.

$$\begin{aligned} &P_x\left(\sup_{0\leqslant s\leqslant t} X_s \leqslant y,\ X_t \in dz\right) \\ =\ & P_x(X_t\in dz) - P_x\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant y,\ X_t \in dz\right) \\ =\ & \frac{1}{z\sigma\sqrt{2\pi t}}e^{-\frac{\left(\ln\left(\frac{z}{x}\right)-\nu\sigma^2 t\right)^2}{2\sigma^2 t}}\,dz - \frac{z^{\nu-1}}{x^{\nu}\sigma\sqrt{2\pi t}}e^{-\frac{\nu^2\sigma^2 t}{2}}e^{-\frac{\left(\ln\frac{y^2}{zx}\right)^2}{2\sigma^2 t}}\,dz \end{aligned}$$

and

$$\begin{aligned} &P_x\left(\sup_{t\leqslant s\leqslant T} X_s \leqslant y,\ X_T \in dz\ \mid X_t = a\right) \\ =\ & P_a\left(\sup_{0\leqslant s\leqslant T-t} X_s \leqslant y,\ X_{T-t} \in dz\right) \\ =\ & \frac{1}{z\sigma\sqrt{2\pi(T-t)}}e^{-\frac{\left(\ln\left(\frac{z}{a}\right)+\frac{\sigma^2}{2}(T-t)\right)^2}{2\sigma^2(T-t)}}\,dz - \frac{\sqrt{a}}{z^{\frac{3}{2}}\sigma\sqrt{2\pi(T-t)}}e^{-\frac{\sigma^2(T-t)}{8}}e^{-\frac{\left(\ln\frac{y^2}{za}\right)^2}{2\sigma^2(T-t)}}\,dz. \end{aligned}$$

■

We finish with the following lemma.

Lemma 3.35

$$\begin{aligned} &P_x\left(\sup_{t\leqslant s\leqslant T} X_s \leqslant y,\ X_T \in dz,\ X_t\in da\right) \\ =\ & \left(\frac{1}{z\sigma\sqrt{2\pi(T-t)}}e^{-\frac{\left(\ln\left(\frac{z}{a}\right)+\frac{\sigma^2}{2}(T-t)\right)^2}{2\sigma^2(T-t)}}\,dz - \frac{\sqrt{a}}{z^{\frac{3}{2}}\sigma\sqrt{2\pi(T-t)}}e^{-\frac{\sigma^2(T-t)}{8}}e^{-\frac{\left(\ln\frac{y^2}{za}\right)^2}{2\sigma^2(T-t)}}\,dz\right) \\ \times\ & \frac{1}{a\sigma\sqrt{2\pi t}}e^{-\frac{\left(\ln\left(\frac{a}{x}\right)+\frac{\sigma^2}{2}t\right)^2}{2\sigma^2 t}}\,da \end{aligned}$$

and

$$
\begin{aligned}
& P_x\left(X_T \in dz,\ X_t \in da \;\middle|\; \sup_{t\leqslant s\leqslant T} X_s \leqslant y\right) \\
= & \left(\frac{1}{z\sigma\sqrt{2\pi\left(T-t\right)}}e^{-\frac{\left(\ln\left(\frac{z}{a}\right)+\frac{\sigma^2}{2}(T-t)\right)^2}{2\sigma^2(T-t)}}\,dz - \frac{\sqrt{a}}{z^{\frac{3}{2}}\sigma\sqrt{2\pi\left(T-t\right)}}e^{-\frac{\sigma^2(T-t)}{8}}e^{-\frac{\left(\ln\frac{y^2}{za}\right)^2}{2\sigma^2(T-t)}}\,dz\right) \\
\times & \frac{1}{a\sigma\sqrt{2\pi t}}e^{-\frac{\left(\ln\left(\frac{a}{x}\right)+\frac{\sigma^2}{2}t\right)^2}{2\sigma^2 t}}\,da\cdot\frac{1}{\mathcal{N}\left(\frac{\ln\left(\frac{y}{x}\right)+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)-\frac{x}{y}\mathcal{N}\left(\frac{-\ln\left(\frac{y}{x}\right)+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)}.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& P_x\left(\sup_{t\leqslant s\leqslant T} X_s \leqslant y,\ X_T \in dz,\ X_t \in da\right) \\
= & P_x\left(\sup_{t\leqslant s\leqslant T} X_s \leqslant y,\ X_T \in dz \mid X_t = a\right)\cdot P_x\left(X_t \in da\right) \\
= & \left(\frac{1}{z\sigma\sqrt{2\pi\left(T-t\right)}}e^{-\frac{\left(\ln\left(\frac{z}{a}\right)+\frac{\sigma^2}{2}(T-t)\right)^2}{2\sigma^2(T-t)}}\,dz - \frac{\sqrt{a}}{z^{\frac{3}{2}}\sigma\sqrt{2\pi\left(T-t\right)}}e^{-\frac{\sigma^2(T-t)}{8}}e^{-\frac{\left(\ln\frac{y^2}{za}\right)^2}{2\sigma^2(T-t)}}\,dz\right) \\
\times & \frac{1}{a\sigma\sqrt{2\pi t}}e^{-\frac{\left(\ln\left(\frac{a}{x}\right)+\frac{\sigma^2}{2}t\right)^2}{2\sigma^2 t}}\,da
\end{aligned}
$$

and

$$
\begin{aligned}
& P_x\left(X_T \in dz,\ X_t \in da \;\middle|\; \sup_{t\leqslant s\leqslant T} X_s \leqslant y\right) \\
= & \frac{P_x\left(\sup_{t\leqslant s\leqslant T} X_s \leqslant y,\ X_T \in dz,\ X_t \in da\right)}{P_x\left(\sup_{t\leqslant s\leqslant T} X_s \leqslant y\right)} \\
= & \left(\frac{1}{z\sigma\sqrt{2\pi\left(T-t\right)}}e^{-\frac{\left(\ln\left(\frac{z}{a}\right)+\frac{\sigma^2}{2}(T-t)\right)^2}{2\sigma^2(T-t)}}\,dz - \frac{\sqrt{a}}{z^{\frac{3}{2}}\sigma\sqrt{2\pi\left(T-t\right)}}e^{-\frac{\sigma^2(T-t)}{8}}e^{-\frac{\left(\ln\frac{y^2}{za}\right)^2}{2\sigma^2(T-t)}}\,dz\right) \\
\times & \frac{1}{a\sigma\sqrt{2\pi t}}e^{-\frac{\left(\ln\left(\frac{a}{x}\right)+\frac{\sigma^2}{2}t\right)^2}{2\sigma^2 t}}\,da\cdot\frac{1}{\mathcal{N}\left(\frac{\ln\left(\frac{y}{x}\right)+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)-\frac{x}{y}\mathcal{N}\left(\frac{-\ln\left(\frac{y}{x}\right)+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)}.
\end{aligned}
$$

■

3.8 Limit distributions

Let $s, p_a \geqslant 0$. Define $A := [0, p_a] \times [0, \infty) \times [0, s]$, $\mathcal{O}^s(t) := \mathcal{O}^s(\{t\}, A)$ and $O^s_\theta(t) := O^s_\theta(\{t\}, A)$. Denote the corresponding intensity measures by $\varpi^s(t)$ and $\varpi^s_\theta(t)$, respectively. Then $\varpi^s_\theta(t) = m^s_\theta \cdot \alpha^s_\theta(E)$ and we use m^s for the mixture of all m^s_θ. Then $\varpi^s(t) = m^s \cdot \alpha^s(E)$. We want to calculate the following probability:

$$
P\left(\mathcal{O}^s(t) = n\right), \qquad n \in \mathbb{N}.
$$

However, $\mathcal{O}^s(t)$ is a mixture of Poisson processes which cannot be computed in a straightforward way. We try to find an asymptotic description of $\mathcal{O}^s(t)$ for t large. Define the standardized order book as

$$\mathcal{O}_{\text{std}}(t) := \frac{\mathcal{O}^s(t) - E(\mathcal{O}^s(t))}{\sqrt{Var(\mathcal{O}^s(t))}}$$

First we calculate

$$E(\mathcal{O}^s(t)) = \int_\Omega \varpi_\theta^s(t)\ dP_\theta = E(\varpi^s(t))$$

and

$$Var(\mathcal{O}^s(t)) = Var(\varpi^s(t)) + E(\mathcal{O}^s(t)).$$

We also look at the standardized intensity measure

$$\varpi_{\text{std}}(t) := \frac{\varpi^s(t) - E(\varpi^s(t))}{\sqrt{Var(\varpi^s(t))}} = \frac{\varpi^s(t) - \int_\Omega \varpi_\theta^s(t)\ dP_\theta}{\sqrt{Var(\varpi^s(t))}}$$

Theorem 3.36 *Assume that* $E(\varpi^s(t)) \to \infty$ *as* $t \to \infty$ *and that*

$$\frac{Var(\varpi^s(t))}{E(\varpi^s(t))} \to c \qquad \text{as} \qquad t \to \infty,$$

where c *is a positive constant. Suppose that* $\frac{\varpi^s(t)-E(\varpi^s(t))}{\sqrt{Var(\varpi^s(t))}}$ *converges in distribution to a random variable* η *as* $t \to \infty$. *Then the standardized order book converges in distribution to*

$$\frac{Z}{\sqrt{1+c}} + \frac{\eta}{\sqrt{1+\frac{1}{c}}},$$

where Z *is a random variable with standard normal distribution, independent of* η.

Proof. As usual, we use characteristic functions to prove this limit theorem. We have to show that

$$\begin{aligned}
\lim_{t\to\infty} \Phi_{\mathcal{O}_{\text{std}}(t)}(u) &= \lim_{t\to\infty} E(\exp\{iu\mathcal{O}_{\text{std}}(t)\}) \\
&= E\left(\exp\left\{iu\frac{Z}{\sqrt{1+c}}\right\}\right) E\left(\exp\left\{iu\frac{\eta}{\sqrt{1+\frac{1}{c}}}\right\}\right) \\
&= E\left(\exp\left\{iu\frac{Z}{\sqrt{1+c}}\right\}\right) E\left(\exp\left\{iu\frac{\eta}{\sqrt{1+\frac{1}{c}}}\right\}\right) \\
&= \exp\left\{-\frac{u^2}{2(1+c)}\right\} E\left(\exp\left\{iu\frac{\eta}{\sqrt{1+\frac{1}{c}}}\right\}\right).
\end{aligned}$$

Therefore we calculate

$$\begin{aligned}
&E\left(\exp\left\{iu\mathcal{O}_{\text{std}}(t)\right\}\right)\\
=\;& E\left(\exp\left\{iu\frac{\mathcal{O}^s(t)-E(\mathcal{O}^s(t))}{\sqrt{Var(\mathcal{O}^s(t))}}\right\}\right)\\
=\;& \exp\left\{-iu\frac{E(\mathcal{O}^s(t))}{\sqrt{Var(\mathcal{O}^s(t))}}\right\}E\left(\exp\left\{iu\frac{\mathcal{O}^s(t)}{\sqrt{Var(\mathcal{O}^s(t))}}\right\}\right)\\
=\;& \exp\left\{-iu\frac{E(\mathcal{O}^s(t))}{\sqrt{Var(\mathcal{O}^s(t))}}\right\}E\left(\exp\left\{\varpi^s(t)\left(e^{\frac{iu}{\sqrt{Var(\mathcal{O}^s(t))}}}-1\right)\right\}\right)\\
=\;& \exp\left\{-iu\frac{E(\mathcal{O}^s(t))}{\sqrt{Var(\mathcal{O}^s(t))}}+E(\mathcal{O}^s(t))\left(e^{\frac{iu}{\sqrt{Var(\mathcal{O}^s(t))}}}-1\right)\right\}\\
&\times E\left(\exp\left\{\frac{\varpi^s(t)-E(\varpi^s(t))}{\sqrt{Var(\varpi^s(t))}}\sqrt{Var(\varpi^s(t))}\left(e^{\frac{iu}{\sqrt{Var(\mathcal{O}^s(t))}}}-1\right)\right\}\right).
\end{aligned}$$

We look at the first term in the above product and consider it, when the time horizon t goes to infinity. We get

$$\begin{aligned}
&\lim_{t\to\infty}\exp\left\{-iu\frac{E(\mathcal{O}^s(t))}{\sqrt{Var(\mathcal{O}^s(t))}}+E(\mathcal{O}^s(t))\left(e^{\frac{iu}{\sqrt{Var(\mathcal{O}^s(t))}}}-1\right)\right\}\\
=\;&\lim_{t\to\infty}\exp\left\{-iu\frac{E(\mathcal{O}^s(t))}{\sqrt{Var(\varpi^s(t))+E(\mathcal{O}^s(t))}}+\frac{E(\mathcal{O}^s(t))\cdot\sqrt{Var(\mathcal{O}^s(t))}}{\sqrt{Var(\varpi^s(t))+E(\mathcal{O}^s(t))}}\left(e^{\frac{iu}{\sqrt{Var(\mathcal{O}^s(t))}}}-1\right)\right\}\\
=\;&\lim_{t\to\infty}\exp\left\{-iu\frac{\sqrt{E(\mathcal{O}^s(t))}}{\sqrt{\frac{Var(\varpi^s(t))}{E(\mathcal{O}^s(t))}+1}}+E(\mathcal{O}^s(t))\left(\exp\left\{\frac{iu}{\sqrt{E(\mathcal{O}^s(t))}\sqrt{\frac{Var(\varpi^s(t))}{E(\mathcal{O}^s(t))}+1}}\right\}-1\right)\right\}\\
=\;&\exp\left(-\frac{1}{2}\frac{u^2}{1+c}\right).
\end{aligned}$$

(Note: $\lim_{x\to\infty}\left(\frac{-iu\sqrt{x}}{\sqrt{1+c}}+x\left(e^{\frac{iu}{\sqrt{x}\sqrt{1+c}}}-1\right)\right)=e^{-\frac{1}{2}\frac{u^2}{1+c}}$). For the second term:

$$\begin{aligned}
&\lim_{t\to\infty}E\left(\exp\left\{\frac{\varpi^s(t)-E(\varpi^s(t))}{\sqrt{Var(\varpi^s(t))}}\sqrt{Var(\varpi^s(t))}\left(e^{\frac{iu}{\sqrt{Var(\mathcal{O}^s(t))}}}-1\right)\right\}\right)\\
=\;&E\left(\lim_{t\to\infty}\exp\left\{\frac{\varpi^s(t)-E(\varpi^s(t))}{\sqrt{Var(\varpi^s(t))}}\sqrt{Var(\varpi^s(t))}\left(e^{\frac{iu}{\sqrt{Var(\mathcal{O}^s(t))}}}-1\right)\right\}\right)\\
=\;&E\left(\lim_{t\to\infty}\exp\left\{\frac{\varpi^s(t)-E(\varpi^s(t))}{\sqrt{Var(\varpi^s(t))}}\sqrt{Var(\varpi^s(t))}\left(e^{\frac{iu}{\sqrt{Var(\varpi^s(t))+E(\mathcal{O}^s(t))}}}-1\right)\right\}\right)\\
=\;&E\left(\lim_{t\to\infty}\exp\left\{\frac{\varpi^s(t)-E(\varpi^s(t))}{\sqrt{Var(\varpi^s(t))}}\sqrt{Var(\varpi^s(t))}\left(e^{\frac{iu}{\sqrt{Var(\varpi^s(t))\left(1+\frac{E(\mathcal{O}^s(t))}{Var(\varpi^s(t))}\right)}}}-1\right)\right\}\right)\\
=\;&E\left(\exp\left\{\frac{iu\eta}{\sqrt{1+\frac{1}{c}}}\right\}\right),
\end{aligned}$$

where we used the bounded convergence theorem as follows:

$$\begin{aligned}
& \left|\exp\left\{\frac{\varpi^s(t)-E(\varpi^s(t))}{\sqrt{Var(\varpi^s(t))}}\sqrt{Var(\varpi^s(t))}\left(e^{\frac{iu}{\sqrt{Var(\mathcal{O}^s(t))}}}-1\right)\right\}\right| \\
= & \left|\exp\left\{(\varpi^s(t)-E(\varpi^s(t)))\left(e^{\frac{iu}{\sqrt{Var(\mathcal{O}^s(t))}}}-1\right)\right\}\right| \\
\leqslant & \exp\left\{-E(\varpi^s(t))\left(\cos\left(\frac{u}{\sqrt{Var(\mathcal{O}^s(t))}}\right)-1\right)\right\} \\
\leqslant & \exp\left(E(\varpi^s(t))\frac{u^2}{2Var(\mathcal{O}^s(t))}\right)\leqslant\exp\left(\frac{1}{2u^2}\right)<\infty.
\end{aligned}$$

(Note: $Var(\varpi^s(t))\geqslant 0$ and $1-\cos x\leqslant\frac{1}{2}x^2$). ■

We now look at

$$Var(\mathcal{O}^s(t))=Var(\varpi^s(t))+E(\mathcal{O}^s(t))$$

and calculate

$$\lim_{t\to\infty}\frac{Var(\varpi^s(t))}{E(\varpi^s(t))}.$$

Then

$$\begin{aligned}
\frac{Var(\varpi^s(t))}{E(\varpi^s(t))} &= \frac{Var(\mathcal{O}^s(t))-E(\mathcal{O}^s(t))}{E(\mathcal{O}^s(t))} \\
&= \frac{Var(\mathcal{O}^s(t))}{E(\mathcal{O}^s(t))}-1.
\end{aligned}$$

Remark 3.37 *If $c=0$, the standardized order book converges in distribution to Z. If $c=\infty$, the standardized order book converges in distribution to η. This can be shown by slightly modifying the previous proof.*

Theorem 3.38 *Assume that α_θ^s is nonrandom, i.e. $\alpha_\theta^s=\alpha^s$ for all θ. Also assume that*

$$\frac{Var(\varpi^s(t))}{E(\varpi^s(t))}\to\infty \qquad \text{as} \qquad t\to\infty$$

and

$$E(\varpi^s(t))\to\infty \qquad \text{as} \qquad t\to\infty.$$

Furthermore, we assume that $\frac{\varpi^s(t)-E(\varpi^s(t))}{\sqrt{Var(\varpi^s(t))}}$ converges in distribution to a random variable η as $t\to\infty$. Then the standardized order book converges in distribution to η.

Proof. Together with the Remark after Theorem 3.36, the result follows. ■

Lemma 3.39 *Assume that $\alpha_\theta^s(t)$ is nonrandom and that the density $f_{(T,R)}(u,v)=\int_0^\infty f_{(T,R,S)}(u,v,w)\,dw$ is continuous. Then*

$$\frac{Var(\varpi^s(t))}{E(\varpi^s(t))}=\frac{Var(m^s\cdot\alpha^s(t))}{E(m^s\cdot\alpha^s(t))}\leqslant\frac{\alpha^s(t)}{E(m^s)}.$$

Proof. The proof follows from clementary observations. ■

Theorem 3.40 *Assume that*

$$\frac{Var\left(\varpi^s\left(t\right)\right)}{E\left(\varpi^s\left(t\right)\right)} \to 0 \qquad \text{as} \qquad t \to \infty$$

and $E\left(\varpi^s\left(t\right)\right) \to \infty$ *as* $t \to \infty$. *Then the standardized order book converges in distribution to a standard normal distribution as* $t \to \infty$.

Proof. The proof is clear from the Remark after Theorem 3.36. ■

We are able to give some results under the condition that $\left(X_t\right)_{t\geqslant 0}$ is a martingale.

Theorem 3.41 *Fix* $p > X_0+2\epsilon$ *for some* $\epsilon > 0$. *Assume that the stock price is a martingale, that* α^s_θ *is nonrandom, i.e.* $\alpha^s_\theta = \alpha^s$ *for all* θ, *that* $f_{(T,R)}$ *is continuous and that*

$$\alpha^s\left([0,t] \times [0,\infty) \times [0,\infty)\right) \to \infty \qquad \text{as } t \to \infty.$$

Then the expected number of orders with absolute price in $[0,p]$ *reaches infinity:*

$$E\left(\varpi^s\left([0,t] \times [0,p) \times [0,\infty)\right)\right) \to \infty \qquad \text{for } t \to \infty.$$

Proof. Set $H := [0,t] \times [0,\infty) \times [0,\infty)$. Then

$$\begin{aligned}
& E\left(\varpi^s\left([0,t] \times [0,p) \times [0,\infty)\right)\right) \\
= \; & E\left(m^s \cdot \alpha^s\left([0,t] \times [0,\infty) \times [0,\infty)\right)\right) \\
= \; & E\left(m^s \cdot \alpha^s\left(H\right)\right) = \alpha^s\left(H\right) \cdot E\left(m^s\right) \\
= \; & \alpha^s\left(H\right) \cdot E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}\left(u,v\right)\, dv\, du\right).
\end{aligned}$$

We try to find a lower bound on

$$E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}\left(u,v\right)\, dv\, du\right)$$

by restricting ourselves to the event

$$\left\{\max_{u\leqslant s\leqslant t} X_s \leqslant X_0 + \epsilon\right\}.$$

Then

$$\max_{u\leqslant s\leqslant t} X_s - X_u \leqslant X_0 + \epsilon - X_u$$

and

$$p - X_u \geqslant X_0 + 2\epsilon - X_u.$$

Therefore

$$p - X_u - \left(\max_{u\leqslant s\leqslant t} X_s - X_u\right) \geqslant X_0 + \epsilon - X_u - \left(X_0 + 2\epsilon - X_u\right) = \epsilon.$$

We assume that $(X_t)_{t\geqslant 0}$ is a martingale. This implies

$$P\left(\sup_{s\geqslant 0} X_s \geqslant X_0+\epsilon\right) \leqslant \frac{\sup_{s\geqslant 0} E(X_s)}{X_0+\epsilon} = \frac{X_0}{X_0+\epsilon}.$$

Then

$$\begin{aligned} P\left(\sup_{u\leqslant s\leqslant t} X_s - X_u \leqslant X_0+\epsilon\right) &\geqslant P\left(\sup_{u\leqslant s\leqslant t} X_s \leqslant X_0+\epsilon\right) \geqslant P\left(\sup_{s\geqslant 0} X_s \leqslant X_0+\epsilon\right) \\ &\geqslant 1-\frac{X_0}{X_0+\epsilon} > 0. \end{aligned}$$

Finally

$$\begin{aligned} & E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) \\ \geqslant\ & P\left(\sup_{u\leqslant s\leqslant t} X_s - X_u \leqslant X_0+\epsilon\right) \int_0^t \int_{X_0+\epsilon-X_u}^{X_0+2\epsilon-X_u} f_{(T,R)}(u,v)\ dv\ du \\ \geqslant\ & \left(1-\frac{X_0}{X_0+\epsilon}\right) \int_0^t \int_{X_0+\epsilon-X_u}^{X_0+2\epsilon-X_u} f_{(T,R)}(u,v)\ dv\ du. \end{aligned}$$

We want

$$X_0+2\epsilon - X_u - X_0 - \epsilon + X_u > 0 \iff \epsilon > 0.$$

■

Remark 3.42 *If $(X_t)_{t\geqslant 0}$ is not a martingale, but has e.g. a positive drift, the above result does not hold. Furthermore, cancellation can prevent the order book from blowing up.*

Theorem 3.43 *Assume in addition that*

$$Var\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) \geqslant c > 0 \qquad \text{for all } t \text{ large enough.}$$

Then

$$\frac{Var(\varpi^s(t))}{E(\varpi^s(t))} \to \infty \qquad \text{as} \qquad t\to\infty.$$

Remark 3.44 *This implies that the standardized order book converges to the same random variable as does the standardized intensity measure.*

Theorem 3.45 *Set $H := ([0,t]\times[0,\infty)\times[0,\infty))$ and define $\alpha^s(t) := \alpha^s(H)$. Then*

$$\begin{aligned} \frac{Var(\varpi^s(t))}{E(\varpi^s(t))} &= \frac{(\alpha^s(t))^2\cdot Var(m^s)}{\alpha^s(t)\,E(m^s)} \\ &= \alpha^s(t)\cdot\frac{Var(m^s)}{E(m^s)} \geqslant \alpha^s(t)\cdot Var(m^s) \to \infty \qquad \text{as} \qquad t\to\infty. \end{aligned}$$

(Note that we implicitly drop the dependence of m^s on t.)

Theorem 3.46 *The assumption*

$$Var\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) \geqslant c > 0 \qquad \text{for all } t \text{ large enough}$$

actually holds if $f_{(T,R)}$ is continuous and $p \geqslant X_0 + \epsilon$ for some $\epsilon > 0$.

Proof. Clearly,

$$\begin{aligned} & P\left(\max_{u\leqslant s\leqslant t} X_s - X_u \geqslant p - \tilde{\epsilon} \text{ and } p - X_u \leqslant p\right) \\ \geqslant\ & P\left(\max_{u\leqslant s\leqslant t} X_s \geqslant p - \tilde{\epsilon} \text{ and } p - X_u \leqslant p\right) \geqslant c_1 > 0 \end{aligned}$$

where c_1 is independent of t $(t > 0)$. Choose $\tilde{\epsilon}$ small enough such that

$$\int_0^\infty \int_{p-\tilde{\epsilon}}^{p} f_{(T,R)}(u,v)\ dv\ du \leqslant E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) - c_2$$

for all t large enough ($t > 0$ is sufficient) and some $c_2 > 0$ (independent of t) (Recall that $E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) \geqslant c$ for some $c > 0$). Then also

$$\int_0^t \int_{p-\tilde{\epsilon}}^{p} f_{(T,R)}(u,v)\ dv\ du \leqslant E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) - c_2$$

(Such an $\tilde{\epsilon}$ exists due to the continuity of $f_{(T,R)}$ and the previous theorem.) Then

$$\begin{aligned} & Var\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right) \\ = \ & E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right)^2 \\ & - \left(E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right)\right)^2 \\ \geqslant\ & \int_{\left\{\max_{u\leqslant s\leqslant t} X_s - X_u \geqslant p - \tilde{\epsilon} \text{ and } p - X_u \leqslant p\right\}} \\ & \left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du - E\left(\int_0^t \int_{\max_{u\leqslant s\leqslant t} X_s - X_u}^{p-X_u} f_{(T,R)}(u,v)\ dv\ du\right)\right)^2 dP \\ \geqslant\ & c_2^2 \cdot P\left(\left\{\max_{u\leqslant s\leqslant t} X_s - X_u \geqslant p - \tilde{\epsilon} \text{ and } p - X_u \leqslant p\right\}\right) \geqslant c_2^2 \cdot c_1 \end{aligned}$$

where c_2 and c_1 are independent of t $(t > 0)$. ■

Theorem 3.47 *Assume that $\alpha^s(t)$ is nonrandom and that $f_{(T,R)}$ is continuous. Then*

$$c \cdot \alpha^s(t) \leqslant \alpha^s(t) \cdot Var(m^s) \leqslant \frac{Var(\varpi^s(t))}{E(\varpi^s(t))} = \frac{Var(m^s \cdot \alpha^s(t))}{E(m^s \cdot \alpha^s(t))} \leqslant \frac{\alpha^s(t)}{E(m^s)} \leqslant \frac{\alpha^s(t)}{c}$$

for some $c > 0$ and all t large enough.

Theorem 3.48 *Assume that*

$$\frac{Var\left(\varpi^{s}\left(t\right)\right)}{E\left(\varpi^{s}\left(t\right)\right)} \rightarrow 0 \qquad \text{as} \qquad t \rightarrow \infty$$

and $E\left(\varpi^{s}\left(t\right)\right) \rightarrow \infty$ *as* $t \rightarrow \infty$. *Then the standardized order book converges in distribution to a standard normal distribution as* $t \rightarrow \infty$.

4 Volume, volatility and the execution probability

This section deals with the relation between traded volume, volume of orders in the book, volatility of the stock price and the execution probability of a limit order. We still work on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P\right)$. The filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ satisfies the usual conditions of right continuity and completeness, $\mathcal{F}_0$ is assumed to be trivial. In this subsection, we allow general semimartingale dynamics for the stock price process $(X_t)_{t\geqslant 0}$, unless otherwise stated. Furthermore we make use of the density of the relative price of arriving sell orders (we will omit the word "sell" in the sequel), denoted by f_t at time t, and say that $f_t(p)$ is the number of arriving orders at price p at time t $(p \geqslant X_t)$. Assume that orders with size greater than one are also counted, assuming that they are split into several orders with size one and included in f_t accordingly. Cancellation is also included.

4.1 The traded volume and the order book

Here we look at the traded volume of orders. We will show, among others, how to find the distribution of the volume, at what price to submit limit orders, if we want a minimum execution probability and the dependence of traded volume on the volatility. Pathwise, we can write down both the total number of orders which are submitted and which are executed. Let $T \in \mathbb{R}$. Then

$$\int_0^T \int_{X_t}^{\infty} f_t(p)\ dp\ dt$$

orders are submitted until time T and

$$\int_0^T \int_{X_t}^{\sup_{t\leqslant s\leqslant T} X_s} f_t(p)\ dp$$

orders are executed. The average volumes are

$$E\left(\int_0^T \int_{X_t}^{\infty} f_t(p)\ dp\ dt\right)$$

and

$$E\left(\int_0^T \int_{X_t}^{\sup_{t\leqslant s\leqslant T} X_s} f_t(p)\ dp\right),$$

respectively.

We do not only want to know the total number of traded orders, but also the volume of orders which are executed and traded up to a limit price p^*. Those numbers are

$$\int_0^T \int_{X_t}^{p^*} f_t(p)\ dp\ dt$$

and

$$\int_0^T \int_{X_t}^{p^*\wedge\sup_{t\leqslant s\leqslant T} X_s} f_t(p)\ dp\ dt,$$

respectively. All those quantities can be calculated numerically, once f_t and X_t are precisely described.

Remark 4.1 *Both the submitted volume and the traded volume give us enough information to fully describe the order book. The number of orders in the book with limit price up to p is just the difference between the orders submitted up to p and those that are executed. Our focus will be on the traded volume, bearing in mind that we can thus entirely characterize the volume of orders in the book.*

4.2 The distribution of traded volume

After characterizing the pathwise volume in the previous section, we will now compute the distribution of traded volume. We first define the properties in which we are interested.

Notation 4.2 (Traded Volume) *We denote the number of orders with limit price up to p, submitted between s and t and executed between s and T by $V_{[s,t]}^{[s,T]}(p)$, $s,t,T \in \mathbb{R}$, $p \in \bar{\mathbb{R}}$, $T \geqslant t > s \geqslant 0$. More precisely*

$$V_{[s,t]}^{[s,T]}(p) = \int_s^t \int_{X_t}^{p \wedge \sup_{t \leqslant s \leqslant T} X_s} f_t(p) \; dp \; dt.$$

As shortcuts, we use

$$\begin{aligned} V_t^{[t,T]}(p) &:= \frac{d}{dt} \int_s^t \int_{X_t}^{p \wedge \sup_{t \leqslant s \leqslant T} X_s} f_t(p) \; dp \; dt, \\ V_{[s,t]}^{[s,T]} &:= V_{[s,t]}^{[s,T]}(\infty), \\ V_t^{[t,T]} &:= V_t^{[t,T]}(\infty). \end{aligned}$$

For an infinite time horizon, we set

$$V_{[s,t]}^{[s,\infty)}(p) := \int_s^t \int_{X_t}^{p \wedge \sup_{t \geqslant s} X_s} f_t(p) \; dp \; dt$$

and similar shortcuts as above. We are interested in the distributions of those quantities.

One of the main tools will be certain quantities related to first-passage times and distributions. For this we will introduce the following notation.

Notation 4.3 *We need to calculate*

$$w_{[t,T]}^{\alpha} := \sup \left\{ \tilde{p} \;\middle|\; P_{X_t}\left(\sup_{t \leqslant s \leqslant T} X_s \geqslant \tilde{p} \right) \geqslant \alpha \right\}$$

and

$$w_{[t,\infty)}^{\alpha} := \sup \left\{ \tilde{p} \;\middle|\; P_{X_t}\left(\sup_{t \leqslant s \leqslant T} X_s \geqslant \tilde{p} \right) \geqslant \alpha \right\}$$

for $t, T \in \mathbb{R}, T \geqslant t \geqslant 0$, $\alpha \in [0,1]$.

We split the task of calculating $V_{[s,t]}^{[s,T]}(p)$ into three parts. First we only look at orders which are submitted at time zero, then at time t and finally at all orders submitted between 0 and T.

4.2.1 The traded volume with orders submitted at 0

Traded volume within a finite time Here we start with the calculation of the distribution of $V_0^{[0,T]}$ and $V_0^{[0,T]}(p)$ where T denotes the finite time horizon. Define a function $h_0 : \mathbb{R} \to [0,1]$ as $h_0(p) = \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}$, where $h_0(p) = 0$ for $p < X_0$. Without assumptions we can give a lower bound for the distribution of $V_0^{[0,T]}$.

Lemma 4.4

$$P\left(V_0^{[0,T]} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) \geqslant P\left(\sup_{0 \leqslant s \leqslant T} X_s \geqslant p\right), \qquad p \in \mathbb{R}$$

and

$$P\left(V_0^{[0,T]} \geqslant \int_{X_0}^{\infty} f_0(\tilde{p})\ d\tilde{p}\right) \geqslant 0.$$

Proof. Observe that

$$\left\{\sup_{0 \leqslant s \leqslant T} X_s \geqslant p\right\} \subset \left\{V_0^{[0,T]} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right\}$$

and

$$\lim_{p \to \infty} P\left(\sup_{0 \leqslant s \leqslant T} X_s \geqslant p\right) = 0.$$

■

The next lemma shows when we actually have equality in the above lemma.

Lemma 4.5 *Assume that $h_0(p)$ is strictly increasing for $p \geqslant X_0$. Then*

$$P\left(V_0^{[0,T]} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) = P\left(\sup_{0 \leqslant s \leqslant T} X_s \geqslant p\right), \qquad p \in \mathbb{R}$$

and

$$P\left(V_0^{[0,T]} \geqslant \int_{X_0}^{\infty} f_0(\tilde{p})\ d\tilde{p}\right) = 0.$$

Proof. $V_0^{[0,T]} \geqslant h_0(p)$ and h_0 strictly increasing imply that $\sup_{0 \leqslant s \leqslant T} X_s \geqslant p$. ■

Remark 4.6 *h_0 is e.g. strictly increasing, if f_0 is strictly positive.*

Theorem 4.7 (Distribution of traded volume) *Assume that h_0 is strictly increasing for $p \in [X_0, c]$ and $h_0(c) = 1$, for some $c \in \mathbb{R}$. Then*

$$P\left(V_0^{[0,T]} \geqslant v\right) = \begin{cases} P\left(\sup_{0 \leqslant s \leqslant T} X_s \geqslant h_0^{-1}(v)\right) & \text{for} \quad 0 \leqslant v < 1 \\ P\left(\sup_{0 \leqslant s \leqslant T} X_s \geqslant c\right) & \text{for} \quad v = 1 \end{cases}$$

Proof. The proof is clear from the previous lemma. Note that $h_0(0) = 0$ and h_0 is continuous. ■

Remark 4.8 *The assumptions in the above theorem are made to reflect empirical data as noted in [BMP02], since they observe that order arrival is truncated at some point.*

Lemma 4.9 *Assume that $h_0(p)$ is strictly increasing for all $p \geqslant X_0$. Then*

$$P\left(V_0^{[0,T]} \geqslant v\right) = \begin{cases} P\left(\sup_{0\leqslant s\leqslant T} X_s \geqslant h_0^{-1}(v)\right) & \text{if } 0 \leqslant v < 1 \\ 0 & \text{if } v = 1 \end{cases}.$$

Remark 4.10 *We always have*

$$P\left(V_0^{[0,T]} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) = P\left(\sup_{0\leqslant s\leqslant T} X_s \geqslant p^*\right)$$

where

$$p^* = \inf\left\{y \;\middle|\; \int_{X_0}^{y} f_0(\tilde{p})\ d\tilde{p} = \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right\}.$$

The distribution of traded volume up to the limit price p^* can now be obtained in a straightforward way.

Theorem 4.11 (Traded volume up to p^*) *Assume that $h_0(p)$ is strictly increasing for $p \in [X_0, c]$ and $h_0(c) = 1$, for some $c \in \mathbb{R}$. Then*

$$P\left(V_0^{[0,T]}(p^*) \geqslant v\right) = \begin{cases} P\left(\sup_{0\leqslant s\leqslant T} X_s \geqslant h_0^{-1}(v)\right) & \text{if } 0 \leqslant v < 1 \text{ and } v \leqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p} \\ P\left(\sup_{0\leqslant s\leqslant T} X_s \geqslant c\right) & \text{if } v = 1 \text{ and } v \leqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p} \\ 0 & \text{if } v > \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p} \end{cases}$$

for $p^* \geqslant X_0$.

Corollary 4.12 (Quantiles of traded volume) *For $\alpha \in [0,1]$, the quantiles of traded volume can be calculated as*

$$P\left(V_0^{[0,T]} \geqslant \int_{X_0}^{w^{\alpha}_{[0,T]}} f_0(\tilde{p})\ d\tilde{p}\right) = \alpha,$$

and

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{w^{\alpha}_{[0,\infty)}} f_0(\tilde{p})\ d\tilde{p}\right) \geqslant \alpha,$$

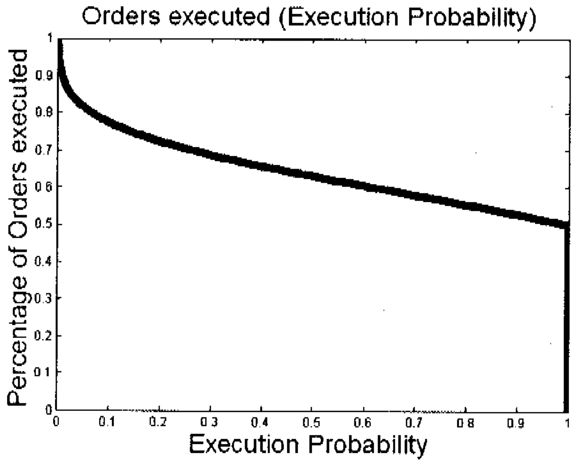
where we recall that

$$w^{\alpha}_{[0,T]} = \sup\left\{\tilde{p} \;\middle|\; P\left(\sup_{0\leqslant s\leqslant T} X_s \geqslant \tilde{p}\right) \geqslant \alpha\right\}$$

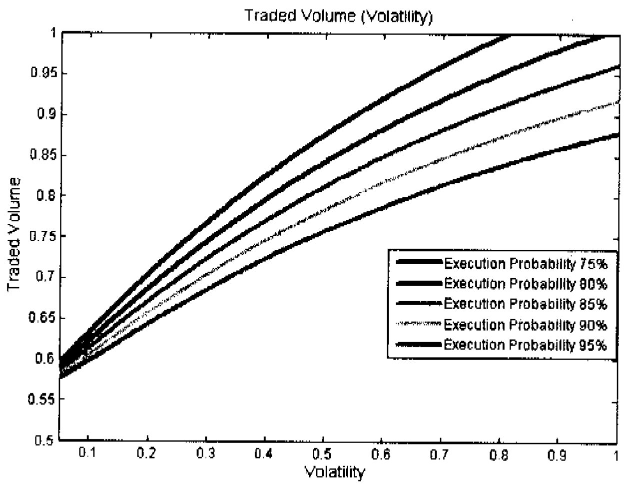
and

$$w^{\alpha}_{[0,\infty)} = \sup\left\{\tilde{p} \;\middle|\; P\left(\sup_{s\geqslant 0} X_s \geqslant \tilde{p}\right) \geqslant \alpha\right\}.$$

The quantiles of traded volume can be visualized as follows: Assume that the stock price is given by a geometric Brownian motion with drift $\mu = 0.08$, volatility $\sigma = 0.2$, time horizon $t = 1$, order arrival parameters $a = 1$, $b = 13$, $\lambda = 1.6$, 50% of orders at the ask (this translates to an execution probability of one in the graph), initial stock price $X_0 = 100$. (These parameters imply that almost all orders arrive between $X_0 = 100$ and $2X_0$.).



Remark 4.13 *We can give a short visualization of traded volume as a function of volatility. Assume that the stock price is given by a geometric Brownian motion with drift* $\mu = 0.08$, *time horizon* $t = 1$, *order arrival parameters* $a = 1$, $b = 13$, $\lambda = 1.6$, 50% *of orders arrive at the ask (which are therefore almost immediately executed), initial stock price* $X_0 = 100$. *(These parameters imply that almost all orders arrive between* $X_0 = 100$ *and* $2X_0$.*) Here we do not consider cancellation.*



Now we go over to the traded volume when the time horizon is infinite.

The traded volume in case of an infinite time horizon Assume for simplicity throughout this subsection that $h_0(p)$ is strictly increasing for $p \geqslant X_0$. The case where h_0 is not strictly increasing can be treated similarly as in Remark (4.10). We obtain the following modifications of Lemma (4.5).

Lemma 4.14 *The distribution of volume, which is ever traded, is given by*

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) = P\left(\sup_{s\geqslant 0} X_s \geqslant p\right), \qquad p \in \mathbb{R}$$

and

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{\infty} f_0(\tilde{p})\ d\tilde{p}\right) = \lim_{p\to\infty} P\left(\sup_{s\geqslant 0} X_s \geqslant p\right).$$

Lemma 4.15 *If $(X_t)_{t\geqslant 0}$ is a continuous positive supermartingale (with $\mathcal{F}_0$ trivial), then*

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) \leqslant \frac{X_0}{p}, \qquad p \in \mathbb{R}$$

and

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{\infty} f_0(\tilde{p})\ d\tilde{p}\right) = 0.$$

Proof. We use the maximal inequality for positive supermartingales:

$$P\left(\sup_{s\geqslant 0} X_s \geqslant p\right) \leqslant \frac{X_0}{p}.$$

■

Lemma 4.16 *If $(X_t)_{t\geqslant 0}$ is a continuous positive martingale (with $\mathcal{F}_0$ trivial) converging $P-$a.s. to zero as t goes to infinity, then*

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) = 1 \wedge \frac{X_0}{p}, \qquad p \in \mathbb{R}$$

and

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{\infty} f_0(\tilde{p})\ d\tilde{p}\right) = 0.$$

Proof. Use the maximal inequality for positive martingales and stop the price process when it first becomes larger than p. ■

Lemma 4.17 *If $(X_t)_{t\geqslant 0}$ is a continuous positive submartingale (with $\mathcal{F}_0$ trivial) such that*

$$P\left(\sup_{s\geqslant 0} X_s \geqslant p\right) = 1, \qquad p \in \mathbb{R}$$

then

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) = 1, \qquad p \in \mathbb{R}$$

and

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{\infty} f_0(\tilde{p})\ d\tilde{p}\right) = 1.$$

Proof. The first part is clear, the second follows by using the continuity from above of the probability measure. ■

Expressions for $V_0^{[0,\infty)}(p^*)$, $p^* \geqslant X_0$, are similar and therefore omitted.

Remark 4.18 *The previous results tell us that an investor who is interested in a long-term execution probability of* α *should not submit orders beyond* $\frac{X_0}{\alpha}$ *if the stock price is a continuous positive martingale converging a.s. to zero as* t *goes to infinity. In that case, he can already achieve an execution probability of* α *if he submits his limit order at* $\frac{X_0}{\alpha}$. *Furthermore, if the stock price process is a continuous positive supermartingale and the investor wants a long-term execution probability of at least* α *he should submit a limit order at price at most* $\frac{X_0}{\alpha}$. *Finally, the last lemma shows under what conditions we can place a limit order as large as possible and still have* $P-$*a.s. execution.*

Now we specialize the stock price even further. Assume that $(X_t)_{t\geqslant 0}$ is given by (2.2). Then we recall the following well-known result [BS02].

Lemma 4.19

$$
\begin{aligned}
& P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant p\right) \\
= \;& 1-\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0}\right)+\left(\frac{\sigma^2}{2}-\mu\right)t}{\sigma\sqrt{t}}\right)+\left(\frac{X_0}{p}\right)^{1-\frac{2\mu}{\sigma^2}}\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0}\right)+\left(\frac{\sigma^2}{2}-\mu\right)t}{\sigma\sqrt{t}}\right).
\end{aligned}
$$

We look at two cases: $\frac{\sigma^2}{2}-\mu>0$ and $\frac{\sigma^2}{2}-\mu\leqslant 0$. The first result is as follows:

Lemma 4.20 *Let* $\frac{\sigma^2}{2}-\mu>0$. *If the desired execution probability is* α, $\alpha\in(0,1]$, *then a limit order should not be placed above* $p=\frac{X_0}{\alpha^d}$, *where* $\frac{1}{d}:=1-\frac{2\mu}{\sigma^2}$.

Proof. If $\frac{\sigma^2}{2}-\mu>0$, then

$$\lim_{t\to\infty} P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant p\right)=\left(\frac{X_0}{p}\right)^{1-\frac{2\mu}{\sigma^2}}.$$

If our desired execution probability is α, we obtain this at a limit price p given by

$$\left(\frac{X_0}{p}\right)^{1-\frac{2\mu}{\sigma^2}}=\alpha.$$

With $\frac{1}{d}=1-\frac{2\mu}{\sigma^2}$, we obtain

$$p=\frac{X_0}{\alpha^d}.$$

This means that eventually the executed volume is not below

$$\int_{X_0}^{\frac{X_0}{\alpha^d}} f_0(\tilde{p})\; d\tilde{p}$$

with probability α. In other words, this also tells us that there is no sense in submitting a limit order beyond $\frac{X_0}{\alpha^d}$ if we want to have an execution probability of at least α. ∎

For the second case we obtain:

Lemma 4.21 *Let* $\frac{\sigma^2}{2}-\mu\leqslant 0$. *If the desired execution probability is* α, $\alpha\in[0,1]$, *we have to wait until* T_α *for our limit order to be executed, where* $P(T_\alpha<\infty)=1$ *and* $T_\alpha=\inf\left\{t\;\middle|\;P\left(\sup_{0\leqslant s\leqslant t}X_s\geqslant p\right)\geqslant\alpha\right\}$.

Proof. If $\frac{\sigma^2}{2} - \mu < 0$ we get

$$\lim_{t\to\infty} P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant p\right) = 1.$$

Therefore, all orders are eventually executed with probability 1. If an execution probability of α is desired, we will have to wait until T_α. We also note that $P(T_\alpha < \infty) = 1$. ■

The influence of volatility We work again with the stock price process given by (2.2). In this subsection, we assume that f_0 is independent of μ and σ^2. It may (and will usually) depend on X_0 though.

First we consider the traded volume as a function of σ^2.

Lemma 4.22 $P\left(V_0^{[0,T]} \geqslant v\right)$ *and* $P\left(V_0^{[0,\infty)} \geqslant v\right)$, $0 \leqslant v \leqslant 1$, *are nondecreasing functions of* σ^2. *If* h_0 *is strictly increasing, they are also strictly increasing.*

Proof. This follows since $P\left(\sup_{0\leqslant s\leqslant T} X_s \geqslant p\right)$ and $P\left(\sup_{s\geqslant 0} X_s \geqslant p\right)$ are increasing functions of σ^2 and therefore both

$$w^\alpha_{[0,T]} = \sup\left\{\tilde{p} \;\middle|\; P\left(\sup_{0\leqslant s\leqslant T} X_s \geqslant \tilde{p}\right) \geqslant \alpha\right\}$$

and

$$w^\alpha_{[0,\infty)} = \sup\left\{\tilde{p} \;\middle|\; P_{X_t}\left(\sup_{s\geqslant t} X_s \geqslant \tilde{p}\right) \geqslant \alpha\right\}$$

are increasing in σ^2 for all $\alpha \in [0,1]$. $P\left(V_0^{[0,T]} \geqslant v\right)$ and $P\left(V_0^{[0,\infty)} \geqslant v\right)$ are thus nondecreasing and actually strictly increasing if h_0 is also strictly increasing. ■

Remark 4.23 *This is a phenomenon which can be observed quite often in real markets. If the volatility in the market increases, the traded volume also increases.*

From now on we assume again that h_0 is strictly increasing.

Lemma 4.24 *For a finite time horizon* T *we have*

$$\lim_{\sigma^2\to\infty} P\left(V_0^{[0,T]} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) = \frac{X_0}{p}, \qquad p \in \mathbb{R}$$

and

$$\lim_{\sigma^2\to\infty} P\left(V_0^{[0,T]} \geqslant \int_{X_0}^{\infty} f_0(\tilde{p})\ d\tilde{p}\right) = 0.$$

Proof. The claim follows by considering

$$\lim_{\sigma^2\to\infty} P\left(\sup_{0\leqslant s\leqslant T} X_s \geqslant p\right) = \frac{X_0}{p}.$$

■

Remark 4.25 *The previous lemma tells us that if σ^2 reaches infinity and all market participants submit their orders only within X_0 and $\frac{X_0}{\alpha}$, i.e. the mass of incoming orders is concentrated on the interval $\left[X_0, \frac{X_0}{\alpha}\right]$, $t \in [0,T]$, then all orders have a chance of at least α to be executed until time T.*

Lemma 4.26 *For an infinite time horizon we get*

$$\lim_{\sigma^2 \to \infty} P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) = \begin{cases} \frac{X_0}{p} & \text{if } \frac{\sigma^2}{2} - \mu > 0 \\ 1 & \text{if } \frac{\sigma^2}{2} - \mu \leqslant 0 \end{cases}, \qquad p \in \mathbb{R}$$

and

$$\lim_{\sigma^2 \to \infty} P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{\infty} f_0(\tilde{p})\ d\tilde{p}\right) = 0.$$

Proof. The claim follows by considering

$$P\left(\sup_{s \geqslant 0} X_s \geqslant p\right) = \begin{cases} \left(\frac{X_0}{p}\right)^{1-\frac{2\mu}{\sigma^2}} & \text{if } \frac{\sigma^2}{2} - \mu > 0 \\ 1 & \text{if } \frac{\sigma^2}{2} - \mu \leqslant 0 \end{cases}$$

and

$$\lim_{\sigma^2 \to \infty} P\left(\sup_{s \geqslant 0} X_s \geqslant p\right) = \begin{cases} \frac{X_0}{p} & \text{if } \frac{\sigma^2}{2} - \mu > 0 \\ 1 & \text{if } \frac{\sigma^2}{2} - \mu \leqslant 0 \end{cases}.$$

■

Remark 4.27 *The previous lemmata tell us that there is no need in submitting limit orders below $\frac{X_0}{\alpha}$ if we are interested in an execution probability of at least α, if either we have a finite time horizon or $\mu < \frac{\sigma^2}{2}$ and the (squared) volatility approaches infinity. Furthermore, they yield immediate upper bounds, uniform in σ^2, on the distributions of $V_0^{[0,T]}$ and $V_0^{[0,\infty)}$. Those are*

$$P\left(V_0^{[0,T]} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) \leqslant \frac{X_0}{p},$$

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) \leqslant \frac{X_0}{p}, \qquad \text{if } \frac{\sigma^2}{2} - \mu > 0, \qquad p \in \mathbb{R}$$

and

$$P\left(V_0^{[0,\infty)} \geqslant \int_{X_0}^{p} f_0(\tilde{p})\ d\tilde{p}\right) = 1 \qquad \text{if } \frac{\sigma^2}{2} - \mu \leqslant 0, \qquad p \in \mathbb{R}$$

4.2.2 The traded volume with orders submitted at t

In this subsection we deal with those orders which are submitted at time t. The goal is to calculate the distributions of $V_t^{[t,T]}$ and $V_t^{[t,T]}(p)$ where T denotes the finite time horizon. The analysis here is very similar to the previous subsection, therefore we will restrict ourselves to the main aspects which are worth mentioning. Define a function $h_t : \Omega \times \mathbb{R} \to [0,1]$ as $h_t(p) = \int_{X_t}^{p} f_t(\tilde{p})\ d\tilde{p}$ with $h_t(p) = 0$ for $p < X_t$. h_t is assumed to be $\mathcal{F}_t$−measurable for every $p \in \mathbb{R}$.

We rewrite Lemma (4.4) and Theorem (4.7).

Lemma 4.28 *Assume that $h_t(p)$ is strictly increasing for $p \geqslant X_t$. Then*

$$P_{X_t}\left(V_t^{[t,T]} \geqslant \int_{X_t}^{p} f_t(\tilde{p})\ d\tilde{p}\right) = P_{X_t}\left(\sup_{t\leqslant s\leqslant T} X_s \geqslant p\right), \qquad p \in \mathbb{R}$$

and

$$P_{X_t}\left(V_t^{[t,T]} \geqslant \int_{X_t}^{\infty} f_t(\tilde{p})\ d\tilde{p}\right) = 0$$

Proof. $V_t^{[t,T]} \geqslant h_t(p)$ and $h_t(p)$ strictly increasing imply that $\sup_{t\leqslant s\leqslant T} X_s \geqslant p$ given X_t. ■

Theorem 4.29 (Distribution of traded volume) *Assume that $h_t(p)$ is strictly increasing for $p \in [X_0, c]$ and $h_t(c) = 1$, $c \in \mathbb{R}$. Then*

$$P_{X_t}\left(V_t^{[t,T]} \geqslant v\right) = \begin{cases} P_{X_t}\left(\sup_{t\leqslant s\leqslant T} X_s \geqslant h_t^{-1}(v)\right) & \text{for} \quad 0 \leqslant v < 1 \\ P_{X_t}\left(\sup_{t\leqslant s\leqslant T} X_s \geqslant c\right) & \text{for} \quad v = 1 \end{cases}$$

for all $v \in [0,1]$.

From now on we assume that $h_t(p)$ is strictly increasing for $p \geqslant X_t$. Next we go over to calculate the average number of orders, which are executed with probability α, i.e.

$$E\left(\int_{X_t}^{w_{[t,T]}^{\alpha}} f_t(\tilde{p})\ d\tilde{p}\right)$$

and characterize the quantiles of $V_t^{[t,T]}$.

Theorem 4.30 *Assume that*

$$\int_{X_t}^{w_{[t,T]}^{\alpha}} f_t(\tilde{p})\ d\tilde{p} = \int_{X_0}^{w_{[0,T-t]}^{\alpha}} f_0(\tilde{p})\ d\tilde{p}, \qquad P - a.s.$$

Then

$$P\left(V_t^{[t,T]} \geqslant \int_{X_0}^{w_{[0,T-t]}^{\alpha}} f_0(\tilde{p})\ d\tilde{p}\right) = \alpha$$

and

$$E\left(\int_{X_t}^{w_{[t,T]}^{\alpha}} f_t(\tilde{p})\ d\tilde{p}\right) = \int_{X_0}^{w_{[0,T-t]}^{\alpha}} f_0(\tilde{p})\ d\tilde{p}$$

for $\alpha \in [0,1]$.

Proof. Note that

$$P_{X_t}\left(V_t^{[t,T]} \geqslant \int_{X_t}^{w_{[t,T]}^{\alpha}} f_t(\tilde{p})\ d\tilde{p}\right) = \alpha$$

and use the assumption. ■

Remark 4.31 *The last theorem gives us the distribution of $V_t^{[t,T]}$. Similarly, we obtain*

$$P\left(V_t^{\infty} \geqslant \int_{X_0}^{w_{[0,\infty)}^{\alpha}} f_0(\tilde{p})\ d\tilde{p}\right) \geqslant \alpha$$

under the condition

$$\int_{X_t}^{w_{[t,\infty)}^{\alpha}} f_t(\tilde{p})\ d\tilde{p} = \int_{X_0}^{w_{[0,\infty)}^{\alpha}} f_0(\tilde{p})\ d\tilde{p}, \qquad P - a.s.$$

If we only look at orders up to the limit price p^*, we obtain

Lemma 4.32

$$P_{X_t}\left(V_t^{[t,T]}(p^*) \geqslant \int_{X_t}^{p} f_t(\tilde{p})\ d\tilde{p}\right) = P_{X_t}\left(\sup_{t\leqslant s\leqslant T} X_s \geqslant p\right), \qquad \text{for } p \leqslant p^*$$

and

$$P_{X_t}\left(V_t^{[t,T]}(p^*) > \int_{X_t}^{p^*} f_t(\tilde{p})\ d\tilde{p}\right) = 0.$$

Proof. Use that

$$V_t^{[t,T]}(p^*) = \int_{X_t}^{\sup_{t\leqslant s\leqslant T} X_s} f_t(\tilde{p})\ d\tilde{p}.$$

■

4.2.3 The traded volume with orders submitted between 0 and T

Finally, we still have to consider the distribution of $V_{[0,T]}^{[0,T]}$. One characterization is given in the next lemma.

Lemma 4.33 *The distribution of* $V_{[0,T]}^{[0,T]}$ *has the following property:*

$$P\left(V_{[0,T]}^{[0,T]} \geqslant \int_0^T \int_{X_t}^{w_{[t,T]}^{\alpha}} f_t(\tilde{p})\ d\tilde{p}\ dt\right) = \alpha, \qquad \text{for } \alpha \in [0,1].$$

Proof. Observe that

$$V_{[0,T]}^{[0,T]} = \int_0^T \int_{X_t}^{\sup_{t\leqslant s\leqslant T} X_s} f_t(\tilde{p})\ d\tilde{p}\ dt.$$

■

Under an additional assumption we obtain more explicit results.

Theorem 4.34 *Let* $\alpha \in [0,1]$. *Assume that*

$$\int_{X_t}^{w_{[t,T]}^{\alpha}} f_t(\tilde{p})\ d\tilde{p} = \int_{X_0}^{w_{[0,T-t]}^{\alpha}} f_0(\tilde{p})\ d\tilde{p}, \qquad P - a.s., \textit{ for all } t \in [0,T].$$

Then

$$P\left(V_{[0,T]}^{[0,T]} \geqslant \int_0^T \int_{X_0}^{w_{[t,T-t]}^{\alpha}} f_0(\tilde{p})\ d\tilde{p}\ dt\right) = \alpha.$$

Proof. Together with the assumption, this follows immediately from the previous lemma. Note that the term $\int_0^T \int_{X_0}^{w_{[t,T-t]}^{\alpha}} f_0(\tilde{p})\ d\tilde{p}\ dt$ is nonrandom. ■

Remark 4.35 *Clearly, the above computations can be helpful in applications. Assume we want to execute a large position of stocks with a certain minimum probability within a certain time horizon. The above results tell us how many such orders will be executed with a certain probability. Comparing this number to our desired volume, we get an intuition for the impact which our activities in the market will have.*

Remark 4.36 (Implied volatility from the order book) *Assume that the stock price process is given as in* (2.2) *. If f_t is strictly positive for all $t \in [0,T]$ and independent of σ^2, then $E\left(V_{[0,T]}^{[0,T]}\right)$ is strictly increasing as a function of σ^2. If we want to infer the volatility of the stock price process by observing the order book, one could proceed as follows: Observe the traded volume in the market and then compute the volatility of the stock price which would lead to the same average volume as the one which was observed in the market.*

4.3 The traded volume and the order arrival

We try to investigate the following empirical results: In Bouchaud et al. [BMP02] it was observed that the density of the distribution of the distance of arriving orders to the current price follows a power-law rule with parameter 1.6. In Plerou et al. [PGG$^+$01] the traded volume was empirically investigated and it was noted that

$$P\left(V_{\Delta t} > v\right) \sim \frac{1}{v^{\lambda}}, \qquad v \in \mathbb{R}_+,$$

where $\lambda = 1.7 \pm 0.1$. $V_{\Delta t}$ denotes the traded volume in an infinitesimal small time interval. At first look, one has a discrepancy between 1.6 and 1.7 and could guess that this is just based on statistical fluctuations or measurement discrepancies, in particular since 1.6 is still in the range given for λ. However, we will show that indeed a value of $\frac{5}{3} \approx 1.67$ follows from theoretical observations if we start with 1.6 for the parameter in the order arrival density. Recall the convention made at the beginning of Section 4 concerning the wording "number of orders".

Start with order submission at time 0 according to the density

$$f_0(p) = \begin{cases} \frac{a}{(b+p)^{\lambda}} & \text{for} \quad a>0,\ \lambda>1,\ b\in\mathbb{R},\, p \geqslant X_0 \\ 0 & \text{for} \quad p < X_0 \end{cases} \tag{4.1}$$

with

$$\int_{X_0}^{\infty} f_0(p)\ dp = 1.$$

In the following we want to look at the situation where we have a fixed time horizon t.

Notation 4.37 *Recall that we denote the total number of orders which are traded in $[s,t]$ and submitted in $[u,v]$ by $V_{[u,v]}^{[s,t]}$, $0 \leqslant s < t$, $0 \leqslant u \leqslant v$. We assume that $V_{[u,v]}^{[s,t]}$ has a density $V_{\tilde{u}}^{\tilde{s}}$ such that*

$$V_{[u,v]}^{[s,t]} = \int_u^v \int_s^t V_{\tilde{u}}^{\tilde{s}}\ d\tilde{u}\ d\tilde{s}.$$

In words, this means that $V_{\tilde{u}}^{\tilde{s}}$ denotes the number of orders which are submitted at time $\tilde{u}$ and traded at time $\tilde{s}$.

Assumption 4.38 *We make the following assumption about homogeneity of order submission and execution*

$$V_{[0,t]}^{t} = \int_0^t V_s^t\ ds = \int_0^t V_0^{t-s}\ ds = V_0^{[0,t]}$$

for all $t \geqslant 0$.

Remark 4.39 *The above assumption is e.g. satisfied under the following condition: Assume that the stock price process* $(X_t)_{t\geqslant 0}$ *and the order arrival* $(f_s)_{s\geqslant 0}$ *are such that*

$$P\left(\int_{X_0}^{\sup_{0\leqslant r\leqslant t-s} X_r} f_0(\tilde{p})\ d\tilde{p} \geqslant x\right) = P\left(\int_{X_s}^{\sup_{s\leqslant r\leqslant t} X_r} f_s(\tilde{p})\ d\tilde{p} \geqslant x\right)$$

for all $x \in \mathbb{R}$. *In words, this could e.g. be satisfied if investors know the distribution of the stock price and then adjust their order submission accordingly. Therefore*

$$V_0^{[0,t-s]} = V_s^{[s,t]}$$

for all $t > s \geqslant 0$. *Then*

$$V_0^{t-s} = V_s^t$$

for $s \geqslant 0$ *and a.e.* t. *This implies that*

$$V_{[0,t]}^t = \int_0^t V_s^t\ ds = \int_0^t V_0^{t-s}\ ds = V_0^{[0,t]}$$

as desired.

Remark 4.40 *Instead of* (4.38) *we can make the slightly stronger assumption that*

$$V_s^t = V_0^{t-s}$$

for all $s \leqslant t$. *In economic terms, this has a very intuitive and appealing meaning: The left-hand side is just the total volume which is submitted at time* s *and executed at time* t. *The right-hand side is the total volume which is submitted at time* 0 *and executed at time* $t - s$. *Therefore, this assumption makes a statement about the behaviour of investors. They are basically interested in the total number of orders which are executed within a certain time horizon. The total orders traded show a time-stationarity effect.*

Theorem 4.41 *Assume that Assumption* (4.38) *holds and the order arrival density is given by* (4.1). *The asymptotic behaviour of traded volume* $V_{[0,t]}^t$ *at time* t *is as follows:*

$$P\left(V_{[0,t]}^t \geqslant v\right) \sim (1-v)^\gamma$$

with $\gamma = \frac{\beta}{\lambda-1}$ *if and only if the stock price process is such that*

$$\lim_{x\to\infty} \frac{P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant x\right)}{\frac{1}{x^\beta}} = c \tag{4.2}$$

for some $0 < c < \infty$ *and* $\beta \in \mathbb{R}$.

Proof. Assume that (4.2) holds. We compute

$$\lim_{v\to 1} \frac{P\left(V_{[0,t]}^t \geqslant v\right)}{(1-v)^\gamma} = \lim_{v\to 1} \frac{P\left(V_{[0,t]}^t \geqslant \int_{X_0}^x f_0(\tilde{p})\ d\tilde{p}\right)}{(1-v)^\gamma} = \lim_{v\to 1} \frac{P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant x\right)}{(1-v)^\gamma}$$

where x is such that

$$v = \int_{X_0}^{x} f_0(\tilde{p})\ d\tilde{p}.$$

The last equal sign follows from our Assumption (4.38). We can solve the formula for v to obtain an equation for x by plugging in $f_0(\tilde{p})$ and integrating:

$$\begin{aligned} v &= \int_{X_0}^{x} f_0(\tilde{p})\ d\tilde{p} = \int_{X_0}^{x} \frac{a}{(b+\tilde{p})^{\lambda}}\ d\tilde{p} \\ &= \frac{1}{-\lambda+1}\frac{a}{(b+\tilde{p})^{\lambda-1}}\Bigg|_{X_0}^{x} \\ &= \frac{1}{-\lambda+1}\frac{a}{(b+x)^{\lambda-1}} + \frac{1}{\lambda-1}\frac{a}{(b+X_0)^{\lambda-1}}. \end{aligned}$$

Using that

$$\int_{X_0}^{\infty} f_0(\tilde{p})\ d\tilde{p} = 1$$

we obtain

$$\frac{1}{\lambda-1}\frac{a}{(b+X_0)^{\lambda-1}} = 1$$

and therefore

$$v = \frac{1}{-\lambda+1}\frac{a}{(b+x)^{\lambda-1}} + 1.$$

This leads to

$$x = \left(\frac{1}{\nu-1}\frac{a}{1-\lambda}\right)^{\frac{1}{\lambda-1}} - b.$$

Then

$$\lim_{v\to 1}\frac{P\left(V_{[0,t]}^{t} \geqslant v\right)}{(1-v)^{\gamma}} = \lim_{v\to 1}\frac{P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant x\right)}{(1-v)^{\gamma}} = \lim_{v\to 1}\frac{\frac{P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant x\right)}{\frac{1}{x^{\beta}}}\frac{1}{x^{\beta}}}{(1-v)^{\gamma}}.$$

Note that $x \to \infty$ for $\nu \to 1$. With $\gamma = \frac{\beta}{\lambda-1}$ and $x = \left(\frac{1}{\nu-1}\frac{a}{1-\lambda}\right)^{\frac{1}{\lambda-1}} - b$ we obtain

$$\begin{aligned} \lim_{v\to 1}\frac{P\left(V_{[0,t]}^{t} \geqslant v\right)}{(1-v)^{\gamma}} &= \lim_{v\to 1}\frac{\frac{P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant x\right)}{\frac{1}{x^{\beta}}}\frac{1}{x^{\beta}}}{(1-v)^{\gamma}} \\ &= c\cdot\lim_{v\to 1}\left(\frac{\frac{1}{\left(\frac{1}{\nu-1}\frac{a}{1-\lambda}\right)^{\frac{1}{\lambda-1}}-b}}{(1-v)^{\frac{1}{\lambda-1}}}\right)^{\beta} = \tilde{c} \end{aligned}$$

for some $0 < \tilde{c} < \infty$, which leads to the desired result. Conversely, if

$$\lim_{v\to 1}\frac{P\left(V_{[0,t]}^{t} \geqslant v\right)}{(1-v)^{\gamma}} = \tilde{c}$$

for some $0 < \tilde{c} < \infty$ we obtain Condition (4.2) by taking into account that

$$\lim_{v \to 1} \frac{\frac{1}{x^\beta}}{(1-v)^\gamma} = c$$

for some $0 < c < \infty$. ∎

Remark 4.42 *Clearly, if we now consider that our total volume is normalized to one and set* $\lambda = 1.6$ *as in [BMP02], we obtain* $\gamma = \frac{5}{3} \in 1.7 \pm 0.1$ *as in [PGG+01]. This shows that both empirical results are linked together, which is something one would not have expected in the beginning.*

It remains to find conditions on the stock price such that (4.2) is valid.

Lemma 4.43 *Assume that* $(X_t)_{t \geqslant 0}$ *is a positive Lévy process, starting at* $X_0 = x_0 \in \mathbb{R}$, *which is heavy-tailed and satisfies*

$$\lim_{x \to \infty} \frac{P(X_t \geqslant x)}{\frac{1}{x^\beta}} = 1 \tag{4.3}$$

for some $\beta \in \mathbb{R}$. *Then* $(X_t)_{t \geqslant 0}$ *fulfills* (4.2).

Proof. The assumption of X being a Lévy process and its heavy-tailedness leads to

$$\lim_{x \to \infty} \frac{P\left(\sup_{0 \leqslant s \leqslant t} X_s \geqslant x\right)}{P(X_t \geqslant x)} = 1.$$

This is shown as follows. Let $(X_t)_{t \geqslant 0}$ be a positive Lévy process and $0 < \tilde{x} < x$. Denote the first entrance time to the interval $[x, \infty)$ by T_x. Then

$$\begin{aligned}
P\left(\sup_{0 \leqslant s \leqslant t} X_s \geqslant x\right) &= P(T_x \leqslant t) \leqslant P(X_t \geqslant x - \tilde{x}) + P(T_x \leqslant t \text{ and } X_t < x - \tilde{x}) \\
&\leqslant P(X_t \geqslant x - \tilde{x}) + P\left(T_x \leqslant t \text{ and } \inf_{0 \leqslant s \leqslant t} (X_{T_x+s} - X_{T_x}) < -\tilde{x}\right) \\
&= P(X_t \geqslant x - \tilde{x}) + P(T_x \leqslant t) \cdot P\left(\inf_{0 \leqslant s \leqslant t} X_s < x_0 - \tilde{x}\right).
\end{aligned}$$

Here we used the independence of T_x and $(X_{T_x+s} - X_{T_x})_{0 \leqslant s \leqslant t}$, the measurability of the inf – function and the stationarity of Lévy process increments. Therefore

$$\begin{aligned}
P\left(\sup_{0 \leqslant s \leqslant t} X_s \geqslant x\right) &\leqslant P(X_t \geqslant x - \tilde{x}) + P\left(\sup_{0 \leqslant s \leqslant t} X_s \geqslant x\right) \cdot P\left(\inf_{0 \leqslant s \leqslant t} X_s < x_0 - \tilde{x}\right) \\
&= P(X_t \geqslant x - \tilde{x}) + P\left(\sup_{0 \leqslant s \leqslant t} X_s \geqslant x\right) \cdot \left(1 - P\left(\inf_{0 \leqslant s \leqslant t} X_s \geqslant x_0 - \tilde{x}\right)\right) \\
&= P(X_t \geqslant x - \tilde{x}) + P\left(\sup_{0 \leqslant s \leqslant t} X_s \geqslant x\right) \\
&\quad - P\left(\sup_{0 \leqslant s \leqslant t} X_s \geqslant x\right) \cdot P\left(\inf_{0 \leqslant s \leqslant t} X_s \geqslant x_0 - \tilde{x}\right)
\end{aligned}$$

which implies

$$P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant x\right)\cdot P\left(\inf_{0\leqslant s\leqslant t} X_s \geqslant x_0-\tilde{x}\right)\leqslant P\left(X_t\geqslant x-\tilde{x}\right).$$

From this we obtain for $\tilde{x}$ sufficiently large such that

$$P\left(\inf_{0\leqslant s\leqslant t} X_s \geqslant x_0-\tilde{x}\right)>0$$

and $x>\tilde{x}$:

$$\limsup_{x\to\infty}\frac{P\left(\sup_{0\leqslant s\leqslant t} X_s\geqslant x\right)}{P\left(X_t\geqslant x\right)}\leqslant\limsup_{x\to\infty}\frac{P\left(X_t\geqslant x-\tilde{x}\right)}{P\left(X_t\geqslant x\right)}\frac{1}{P\left(\inf_{0\leqslant s\leqslant t} X_s\geqslant x_0-\tilde{x}\right)}.$$

Using the heavy-tailedness of $(X_t)_{t\geqslant 0}$ and letting $\tilde{x}\to x_0$ gives

$$\limsup_{x\to\infty}\frac{P\left(\sup_{0\leqslant s\leqslant t} X_s\geqslant x\right)}{P\left(X_t\geqslant x\right)}\leqslant 1.$$

The other inequality is always satisfied, so this leads immediately to

$$\limsup_{x\to\infty}\frac{P\left(\sup_{0\leqslant s\leqslant t} X_s\geqslant x\right)}{P\left(X_t\geqslant x\right)}=1.$$

Together with (4.3), the following calculations hold true:

$$\begin{aligned}&\lim_{x\to\infty}\frac{P\left(\sup_{0\leqslant s\leqslant t} X_s\geqslant x\right)}{\frac{1}{x^\beta}}\\ =&\lim_{x\to\infty}\frac{P\left(\sup_{0\leqslant s\leqslant t} X_s\geqslant x\right)}{P\left(X_t\geqslant x\right)}\frac{P\left(X_t\geqslant x\right)}{\frac{1}{x^\beta}}=1\end{aligned}$$

For a similar result in the case of processes not necessarily positive and starting at zero and for further results in that direction as well as the tail equivalence of $\sup_{0\leqslant s\leqslant t} X_s$, X_t and the right tail of the Lévy measure of Lévy processes see among others [RS93], [BMS02], [Asm98], [Wil87], [Bra05], [HL05a], [HLMS05] and [HL05b]. ■

Remark 4.44 *Note that we did not include cancellation here. However, assuming e.g. some kind of uniform cancellation over all orders (independent of the limit price), the results from above do not change.*

Under Assumption (4.38) the result in (4.41) also holds for $\beta=1$, if the stock price process $(X_t)_{t\geqslant 0}$ is a continuous positive martingale (with $\mathcal{F}_0$ trivial) converging a.s. to zero as t goes to infinity and we use the approximation that

$$V_0^{[0,t]}\approx V_0^{[0,\infty)}.$$

In that case, we can use the fact that

$$\lim_{x\to\infty}\frac{P\left(\sup_{s\geqslant 0} X_s\geqslant x\right)}{\frac{1}{x}}=c$$

for some $0<c<\infty$. Here we do not need to refer to Lemma (4.43).

5 The time horizon of investors

Here we use a simple model to get a quantitative answer about the time horizons of investors using information which we obtain from the limit-order book and the order arrival process.

When an investor invests in the stock markets using the limit-order book, i.e. buying or selling shares with the help of limit orders, he might be interested in many different objectives: maximize his utility, selling a large amount of stocks rapidly, selling or buying under certain other constraints, etc. After solving his individual optimization problems, he must end up with a number which tells him how many orders to invest at what limit price, considering his time horizon. Ultimately, he has only two aspects to take into account: What is the probability that his order is executed (**execution probability**) and what time-frame (**time horizon**) is he looking at? We show how we can infer those two quantities from the order book. Even if investors have different reasons for submitting orders, every submitted order can still be translated into and fully characterized by the concept of execution probability and time horizon.

5.1 Infinite time horizon and the distribution of the long-term execution probabilities

We start with the long-term execution probability, using an infinite time horizon.

Definition 5.1 (Execution probability) *We define the* ***execution probability*** $\mathcal{P}_E^\infty$ *of a limit order with price p, considering an infinite time horizon, with the help of the following distribution, as*

$$\begin{aligned} P_{\mathcal{P}_E^\infty} &: \mathbb{R} \to [0,1] \\ \alpha &\longmapsto \int_{X_0}^{w^\alpha_{[0,\infty)}} f_0(\tilde{p})\ d\tilde{p} = P\left(\mathcal{P}_E^\infty \geqslant \alpha\right), \qquad \text{for } \alpha \in (0,1] \\ P\left(\mathcal{P}_E^\infty \geqslant \alpha\right) &= 1, \qquad \text{for } \alpha \leqslant 0, \\ P\left(\mathcal{P}_E^\infty > 1\right) &= 0. \end{aligned}$$

Lemma 5.2 *Assume that* $(X_t)_{t\geqslant 0}$ *is a continuous local martingale such that* $\langle X,X\rangle_\infty = \infty$ $P-$*a.s. Then* $\mathcal{P}_E^\infty$ *has a one-point distribution.*

Proof. The assumptions imply that

$$\overline{\lim_{t\to\infty}} X_t = \infty, \qquad P-\text{a.s.}$$

(see Revuz-Yor, chapter V [RY04]) Then $w^\alpha_{[0,\infty)} = \infty$ $P-$ a.s. for $\alpha \in (0,1]$ and therefore $P\left(\mathcal{P}_E^\infty \geqslant \alpha\right) = 1$ for $\alpha \leqslant 1$. ∎

Lemma 5.3 *Assume that* $(X_t)_{t\geqslant 0}$ *is a continuous positive martingale (with* $\mathcal{F}_0$ *trivial) converging* $P-$ *a.s. to zero as t goes to infinity. Then*

$$P\left(\mathcal{P}_E^\infty \geqslant \alpha\right) = \begin{cases} \int_{X_0}^{\frac{X_0}{\alpha}} f_0(\tilde{p})\ d\tilde{p} & \text{for } \alpha \in (0,1] \\ 1 & \text{for } \alpha = 0 \end{cases}.$$

Proof. The claim follows immediately by observing that

$$P\left(\sup_{s\geqslant 0} X_s \geqslant p\right) = 1 \wedge \frac{X_0}{p}.$$

■

We drop the martingale assumption but restrict ourselves to the case of geometric Brownian motion and, as usual, we differentiate two cases, $\mu < \frac{\sigma^2}{2}$ and $\mu \geqslant \frac{\sigma^2}{2}$. The second case is trivial, since we would always have an execution probability equal to one. We look at the first case only.

Lemma 5.4 *Assume that $(X_t)_{t\geqslant 0}$ is given as in (2.2) with $\mu < \frac{\sigma^2}{2}$. Then*

$$P\left(\mathcal{P}_E^\infty \geqslant \alpha\right) = \int_{X_0}^{X_0\cdot\alpha^{\frac{\sigma^2}{2\mu-\sigma^2}}} f_0\left(\tilde{p}\right)\, d\tilde{p}, \qquad \text{for } \alpha \in (0,1].$$

Proof. This follows from the distribution of the first-passage time for geometric Brownian motion. ■

Remark 5.5 *Of course, including cancellation can resolve the issue with the positive probability of having infinite time horizons.*

5.2 The distribution of the time horizon of limit order traders

Here we introduce the time horizon of an investor which submits a limit order at price p and the distribution of the time horizons of all limit order traders which submit orders at time 0.

Definition 5.6 (Time horizon) *We denote the first passage time of the limit price p as $\mathcal{T}_H^p : \mathbb{R} \to \bar{\mathbb{R}}$ with*

$$\mathcal{T}_H^p := \inf\{t \mid X_t \geqslant p\}$$

and call it the **p−*time horizon***. *Assume that the order arrival density (distance from X_0) is given by f_0. Then we define the* ***time horizon*** *of investors $\mathcal{T}_H$ as $\mathcal{T}_H : \mathbb{R} \to \bar{\mathbb{R}}$ with the distribution*

$$P\left(\mathcal{T}_H \leqslant t\right) = \int_0^\infty P\left(\mathcal{T}_H^p \leqslant t\right)\cdot f_0\left(p\right)\, dp.$$

We want to include cancellation here.

Notation 5.7 (Time of cancellation) *We denote the* ***time of cancellation*** *of a limit order submitted at $t = 0$ with price p by τ^p, $p \geqslant X_0$. This means that the order is cancelled at τ^p if it has not been executed earlier.*

Definition 5.8 (Time horizon with cancellation) *If orders are cancelled at time τ^p, the* **p−*time horizon with cancellation*** *is defined as*

$$\mathcal{T}_{H^c}^p := \min\{\tau^p, \mathcal{T}_H^p\}$$

and for the ***time horizon with cancellation***, *denoted by $\mathcal{T}_{H^c}$, we replace $\mathcal{T}_H^p$ by $\mathcal{T}_{H^c}^p$.*

In order to explicitly calculate the time horizon we make two assumptions. First the stock price is given as in (2.2) and we set $\nu := \frac{\mu}{\sigma^2} - \frac{1}{2}$. Second, we assume that τ^p is independent of $(X_t)_{t \geqslant 0}$ (Note however, that the cancellation time can still depend on p.).

We compute the distribution of the time horizon both with and without cancellation. Recall that the order arrival density f_0 is given by $f_0(p) = \frac{a}{(b+p)^\lambda}$ for $a, \lambda > 0$, $b \in \mathbb{R}$, $p \geqslant X_0$ such that $\int_{X_0}^{\infty} f_0(p)\ dp = 1$. If $\mu \leqslant \frac{\sigma^2}{2}$ we obtain that the $p-$time horizon is infinite with probability $\frac{X_0}{p}$ if there is no cancellation.

Theorem 5.9 (Time horizon) *Assume that $\mu > \frac{\sigma^2}{2}$ and no cancellation of orders occurs. Then the distribution of the time horizons of investors which submit limit orders is given by*

$$
\begin{aligned}
&P(\mathcal{T}_H \leqslant t) \\
&= \int_{X_0}^{\infty} \left[\int_0^t \frac{\ln \frac{p}{X_0}}{\sigma\sqrt{2\pi} s^{\frac{3}{2}}} \left(\frac{p}{X_0}\right)^{\nu} \exp\left\{ \frac{-\nu^2\sigma^2 s}{2} - \frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 s} \right\} ds \right] \cdot \frac{a}{(b+p)^{\lambda}}\ dp.
\end{aligned}
$$

Proof. We can compute

$$
P(\mathcal{T}_H^p \in dt) = \frac{\left|\ln \frac{p}{X_0}\right|}{\sigma\sqrt{2\pi} t^{\frac{3}{2}}} \left(\frac{p}{X_0}\right)^{\nu} \exp\left\{ \frac{-\nu^2\sigma^2 t}{2} - \frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 t} \right\} dt
$$

and

$$
P(\mathcal{T}_H^p = \infty) = \begin{cases} 1 - \left(\frac{X_0}{p}\right)^{|\nu|-\nu} & X_0 \leqslant p \\ 1 - \left(\frac{X_0}{p}\right)^{|\nu|+\nu} & p \leqslant X_0 \end{cases}.
$$

For $\mu > \frac{\sigma^2}{2}$ and $X_0 \leqslant p$ we obtain $P(\mathcal{T}_H^p = \infty) = 0$. Together with

$$
P(\mathcal{T}_H \leqslant t) = \int_{X_0}^{\infty} P(\mathcal{T}_H^p \leqslant t) \cdot f_0(p)\ dp
$$

the results follows. ■

Theorem 5.10 (Time horizon with cancellation) *Assume that $\mu > \frac{\sigma^2}{2}$ and that the order with limit price $p \geqslant X_0$ is cancelled at time τ^p, if it has not been executed earlier. The time horizon $\mathcal{T}_{H^c}$ of investors is then*

$$
\begin{aligned}
&P(\mathcal{T}_{H^c} \leqslant t) \\
&= 1 - \int_{X_0}^{\infty} \left[\int_t^{\infty} \frac{\ln \frac{p}{X_0}}{\sigma\sqrt{2\pi} s^{\frac{3}{2}}} \left(\frac{p}{X_0}\right)^{\nu} e^{\left\{ \frac{-\nu^2\sigma^2 s}{2} - \frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 s} \right\}} ds \right] \cdot P(\tau^p \geqslant t) \cdot \frac{a}{(b+p)^{\lambda}}\ dp.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& P\left(\min\left\{\mathcal{T}_H^p, \tau^p\right\} \leqslant t\right) \\
= \ & P\left(\mathcal{T}_H^p \leqslant t \text{ or } \tau^p \leqslant t\right) \\
= \ & 1 - P\left(\mathcal{T}_H^p \geqslant t\right) \cdot P\left(\tau^p \geqslant t\right) \\
= \ & 1 - \int_t^\infty \frac{\left|\ln \frac{p}{X_0}\right|}{\sigma\sqrt{2\pi}s^{\frac{3}{2}}} \left(\frac{p}{X_0}\right)^\nu \exp\left\{\frac{-\nu^2\sigma^2 s}{2} - \frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 s}\right\} \cdot P\left(\tau^p \geqslant t\right)\ ds
\end{aligned}
$$

with the help of the independence of τ^p and $(X_t)_{t\geqslant 0}$. Then

$$
\begin{aligned}
& P\left(\mathcal{T}_{H^c} \leqslant t\right) \\
= \ & \int_{X_0}^\infty P\left(\min\left\{\mathcal{T}_H^p, \tau^p\right\} \leqslant t\right) \cdot f_0(p)\ dp \\
= \ & \int_{X_0}^\infty \left[1 - P\left(\mathcal{T}_H^p \geqslant t\right) \cdot P\left(\tau^p \geqslant t\right)\right] \cdot f_0(p)\ dp \\
= \ & 1 - \int_{X_0}^\infty P\left(\mathcal{T}_H^p \geqslant t\right) \cdot P\left(\tau^p \geqslant t\right) \cdot f_0(p)\ dp \\
= \ & 1 - \int_{X_0}^\infty \left[\int_t^\infty \frac{\ln \frac{p}{X_0}}{\sigma\sqrt{2\pi}s^{\frac{3}{2}}} \left(\frac{p}{X_0}\right)^\nu e^{\left\{\frac{-\nu^2\sigma^2 s}{2} - \frac{\left(\ln\left(\frac{p}{X_0}\right)\right)^2}{2\sigma^2 s}\right\}} ds\right] \cdot P\left(\tau^p \geqslant t\right) \cdot f_0(p)\ dp.
\end{aligned}
$$

Further note that

$$
f_0(p) = \frac{a}{(b+p)^\lambda}, \qquad p \geqslant X_0.
$$

■

Remark 5.11 *Observe that we also get a result for $\mu > \frac{\sigma^2}{2}$, if cancellation is included. This leads to $P\left(\mathcal{T}_{H^c} < \infty\right) = 1$ if $P\left(\tau^p < \infty\right) = 1$.*

Note that we allow here a cancellation rate which may depend on the limit price. Potters and Bouchaud [BMP02] already empirically investigated that the life-time of a given order increases as one moves away from the bid-ask. They claim that any theory of the order book should include such a non-uniform cancellation rate.

We can also calculate various moments of the time horizon, conditional on it being finite, using

$$
\begin{aligned}
& E\left(e^{-\alpha\mathcal{T}_H^p}\right) := E\left(e^{-\alpha\mathcal{T}_H^p}; \mathcal{T}_H^p < \infty\right) \\
= \ & \begin{cases} \left(\frac{X_0}{p}\right)^{\sqrt{\nu^2+\frac{2\alpha}{\sigma^2}}-\nu} & \text{for } X_0 \leqslant p \\ \left(\frac{p}{X_0}\right)^{\sqrt{\nu^2+\frac{2\alpha}{\sigma^2}}+\nu} & \text{for } p \leqslant X_0 \end{cases}
\end{aligned}
\tag{5.1}
$$

The expected time horizon of an investor, which submits an order at price p, is given in the next corollary.

Corollary 5.12 *Assume that* $\mu > \frac{\sigma^2}{2}$, $p \geqslant X_0$ *and that there is no cancellation. Then*

$$E\left(\mathcal{T}_H^p\right) = \ln\left(\frac{p}{X_0}\right)\frac{1}{\nu\sigma^2}$$

and

$$\mathrm{Var}\left(\mathcal{T}_H^p\right) = \ln\left(\frac{p}{X_0}\right)\cdot\frac{1}{\nu^3\sigma^4}.$$

Proof. We use (5.1) with $p \geqslant X_0$ and compute

$$E\left(\mathcal{T}_H^p; \mathcal{T}_H^p < \infty\right)^k = (-1)^k \left.\frac{d^k\, E\left(e^{-\alpha\mathcal{T}^p}\right)}{d\,\alpha^k}\right|_{\alpha=0}.$$

Therefore

$$\begin{aligned} E\left(\mathcal{T}_H^p; \mathcal{T}_H^p < \infty\right) &= -\frac{d\, E\left(e^{-\alpha\mathcal{T}^p}\right)}{d\,\alpha}\Big|_{\alpha=0} \\ &= -\frac{d}{d\alpha}\left. e^{\ln\left(\frac{X_0}{p}\right)\left(\sqrt{\nu^2+\frac{2\alpha}{\sigma^2}}-\nu\right)}\right|_{\alpha=0} \\ &= -e^{\ln\left(\frac{X_0}{p}\right)\left(\sqrt{\nu^2+\frac{2\alpha}{\sigma^2}}-\nu\right)}\cdot\ln\left(\frac{X_0}{p}\right)\frac{1}{2\sqrt{\nu^2+\frac{2\alpha}{\sigma^2}}}\cdot\left.\frac{2}{\sigma^2}\right|_{\alpha=0}. \end{aligned}$$

Assume that $\mu > \frac{\sigma^2}{2}$, in other words, $\nu > 0$. Then $P\left(\mathcal{T}_H^p < \infty\right) = 1$ and

$$E\left(\mathcal{T}_H^p\right) = \ln\left(\frac{p}{X_0}\right)\frac{1}{\nu\sigma^2}.$$

For the second moment we calculate

$$\begin{aligned} E\left(\mathcal{T}_H^p\right)^2 &= \left.\frac{d^2\, E\left(e^{-\alpha\mathcal{T}^p}\right)}{d\,\alpha^2}\right|_{\alpha=0} \\ &= \ln\left(\frac{X_0}{p}\right)\cdot\frac{1}{\sigma^2}\cdot\left(\ln\left(\frac{X_0}{p}\right)\frac{1}{\nu^2}\frac{1}{\sigma^2}-\frac{1}{\nu^3}\frac{1}{\sigma^2}\right) \\ &= \ln\left(\frac{X_0}{p}\right)\cdot\frac{1}{\sigma^4}\cdot\left(\ln\left(\frac{X_0}{p}\right)\frac{1}{\nu^2}-\frac{1}{\nu^3}\right). \end{aligned}$$

Then

$$\begin{aligned} Var\left(\mathcal{T}_H^p\right) &= \ln\left(\frac{X_0}{p}\right)\cdot\frac{1}{\sigma^4}\cdot\left(\ln\left(\frac{X_0}{p}\right)\frac{1}{\nu^2}-\frac{1}{\nu^3}\right)-\left(\ln\left(\frac{p}{X_0}\right)\frac{1}{\nu\sigma^2}\right)^2 \\ &= \ln\left(\frac{p}{X_0}\right)\cdot\frac{1}{\nu^3\sigma^4}. \end{aligned}$$

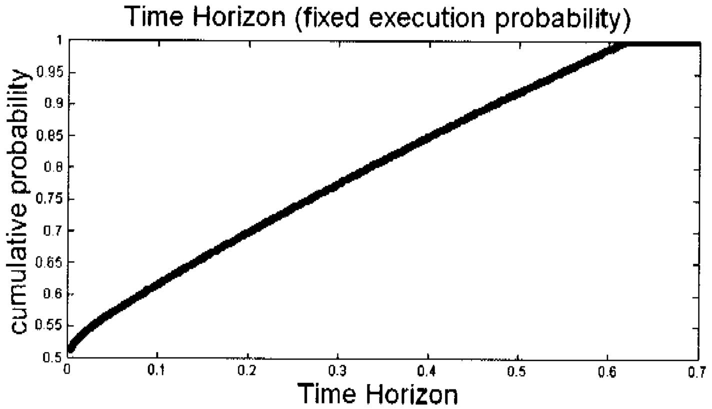
■

Under the assumptions of Corollary (5.12) the expected time horizon can also be calculated in a straightforward way and is given by

$$E(\mathcal{T}_H) = \frac{a\left(X_0+b\right)}{\nu\sigma^2\left(X_0+b\right)^{\lambda}(\lambda-1)}\left(\ln\left(X_0^2+bX_0\right)+\frac{1}{\lambda-1}\right).$$

Remark 5.13 *Lynch and Zumbach [LZ03] describe four types of investors with different time horizons: Intra-day market makers and arbitrageurs, hedge funds and active portfolio managers, passive portfolio managers, and finally the central banks and pension funds. They analyze the relation between historical volatility and realized volatility and find that the time horizons of the above four groups are essentially as follows:* $2-4$ *hours,* 1 *day,* 1 *week and* $2-5$ *weeks. They emphasize that their main point is the existence of investors with different time horizons, not the precise identification and characterization of each group.*

If we visualize the distribution of time horizon with a fixed execution probability of 80%, we obtain the following: Assume that the stock price is given by a geometric Brownian motion with drift $\mu = 0.08$, volatility $\sigma = 0.2$, the order arrival parameters are $a = 13$, $b = 1$, $\lambda = 1.6$, 50% of orders arrive at the ask price (This translates to a time horizon very close to zero in the graph.).



For illustrative purposes, we look at one special case. Assume that all orders in the market arrive with a fixed, **exogenously given**, desired execution probability of α. Based on this we can calculate the time horizons of investors which results from the order arrival distribution.

Notation 5.14 (Time horizon with given α) *We use the notation* ${}_{\alpha}\mathcal{T}_H$ *for the distribution of time horizons, provided that all orders arrive with a desired execution probability of* α.

Then

$$P\left({}_{\alpha}\mathcal{T}_H \leqslant t\right) = \int_{X_0}^{w_{[0,t]}^{\alpha}} f_0\left(\tilde{p}\right)\, d\tilde{p}$$

where we recall that $w_{[0,t]}^{\alpha}$ denotes the solution of

$$\sup\left\{p \;\middle|\; P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant p\right) = \alpha\right\}.$$

Remark 5.15 *Harris and Hasbrouck [HH96a] note that* 82% *of the limit orders in their NYSE TORQ database are day-orders (an order which can be filled anytime until the market closes), although the number of good-till canceled limit orders is also substantial, about* 17% *in their sample.*

5.3 The execution probability

In the last subsection we calculated the distribution of time horizons, $\mathcal{T}_H^p$ and $\mathcal{T}_H$. Now we look at the distribution of the execution probabilities, assuming that time horizons are given by $\mathcal{T}_H^p$ and $\mathcal{T}_H$. We also assume that the $p-$time horizon $\mathcal{T}_H^p$ is almost surely finite. The case of a positive probability of an infinite time horizon can be dealt with using the methods of subsection (5.1).

Definition 5.16 ($p-$Execution Probability) *If the $p-$time horizon is given by $\mathcal{T}_H^p$, denote the (induced)* **p−*execution probability*** *by $\mathcal{P}_E^{\mathcal{T}_H^p}$. The execution probability is defined by means of its distribution:*

$$P\left(\mathcal{P}_E^{\mathcal{T}_H^p} \leqslant \alpha\right) = \begin{cases} P_{\mathcal{T}_H^p}\left([0,t_\alpha]\right) & \text{for} \quad \alpha \in [0,1] \\ 0 & \text{for} \quad \alpha \leqslant 0 \\ 1 & \text{for} \quad \alpha \geqslant 1 \end{cases}.$$

$P_{\mathcal{T}_H^p}$ denotes the probability distribution on the time interval $[0,\infty)$ which is induced by the random time horizon $\mathcal{T}_H^p$. t_α is defined as

$$t_\alpha = \sup\left\{t \;\middle|\; P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant p\right) \leqslant \alpha\right\}.$$

Definition 5.17 (Execution Probability) *The* ***execution probability*** *of limit order traders is defined as $\mathcal{P}_E^{\mathcal{T}_H}$ with*

$$P\left(\mathcal{P}_E^{\mathcal{T}_H} \leqslant \alpha\right) = \int_0^\infty P\left(\mathcal{P}_E^{\mathcal{T}_H^p} \leqslant \alpha\right)\cdot f_0\left(p\right)\, dp, \qquad \alpha \in \mathbb{R}.$$

Theorem 5.18 *Both $\mathcal{P}_E^{\mathcal{T}_H^p}$ and $\mathcal{P}_E^{\mathcal{T}_H}$ have a uniform distribution on $[0,1]$.*

Proof. Let $\alpha \in [0,1]$. We compute

$$\begin{aligned} P\left(\mathcal{P}_E^{\mathcal{T}_H^p} \leqslant \alpha\right) &= P_{\mathcal{T}_H^p}\left([0,t_\alpha]\right) \\ &= P\left(\mathcal{T}_H^p \leqslant t_\alpha\right) = P\left(\sup_{0\leqslant s\leqslant t_\alpha} X_s \geqslant p\right) = \alpha. \end{aligned}$$

The uniform distribution of $\mathcal{P}_E^{\mathcal{T}_H}$ follows immediately. ■

Remark 5.19 *If we include cancellation, we have to replace $\mathcal{T}_H^p$ by $\min\{\mathcal{T}_H^p, \tau^p\}$. More precisely, replace t_α by*

$$t_\alpha^c = \sup\left\{t \;\middle|\; P\left(\sup_{0\leqslant s\leqslant t} X_s \geqslant p \text{ or } \tau^p \leqslant t\right) \leqslant \alpha\right\}.$$

Then we compute

$$\begin{aligned} P\left(\mathcal{P}_E^{\mathcal{T}_{H^c}^p} \leqslant \alpha\right) &= P_{\mathcal{T}_{H^c}^p}\left([0,t_\alpha^c]\right) \\ &= P\left(\mathcal{T}_{H^c}^p \leqslant t_\alpha^c\right) = P\left(\min\{\mathcal{T}_H^p, \tau^p\} \leqslant t_\alpha^c\right) \\ &= P\left(\sup_{0\leqslant s\leqslant t_\alpha^c} X_s \geqslant p \text{ or } \tau^p \leqslant t_\alpha^c\right) = \alpha. \end{aligned}$$

The uniform distributions of $\mathcal{P}_E^{\mathcal{T}_{H^c}^p}$ and $\mathcal{P}_E^{\mathcal{T}_{H^c}}$ follow immediately.

Again, we look at one special case. We assume now that the time horizon is **exogenously** given and all orders in the market arrive with a fixed time horizon T. Based on this we can also calculate the execution probabilities of investors which results from the order arrival distribution.

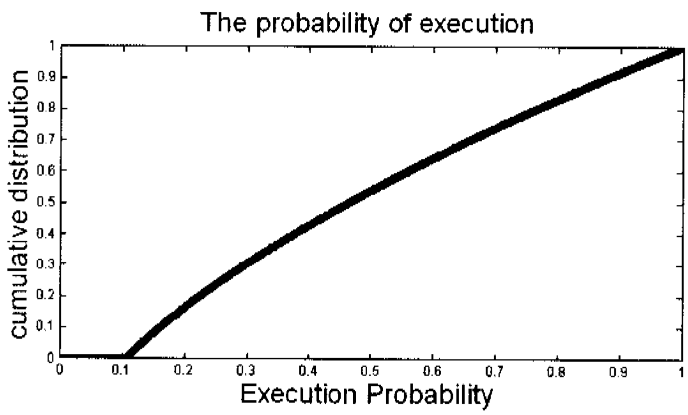
Notation 5.20 (Execution probability with fixed time horizon) *We use the notation $\mathcal{P}_E^T$ for the distribution of execution probabilities, provided that all orders have a time horizon T.*

Then

$$P\left(\mathcal{P}_E^T \geqslant \alpha\right) = \int_{X_0}^{w_{[0,T]}^{\alpha}} f_0\left(\tilde{p}\right)\ d\tilde{p}.$$

In case of a geometric Brownian motion with $\mu < \frac{\sigma^2}{2}$ and $T = \infty$, this distribution was analyzed earlier in subsection (5.1).

A short visualization with the parameters $\mu = 0$, $\sigma = 0.2$, $a = 13$, $b = 1$, $\lambda = 1.6$ and an infinite time horizon leads to (The zero cumulative distribution for an execution probability of less than 11% comes from the fact that in case of an infinite time horizon, $w_{[0,\infty)}^{\alpha}$ takes a value such that $\int_{X_0}^{w_{[0,\infty)}^{\alpha}} f_0\left(\tilde{p}\right)\ d\tilde{p} = 1$.):



Note that this behaviour shows among others, that orders are not placed so far away from the current stock price that they have practically no chance of being executed. This comes from our choice for the parameters a, b and λ since they imply that basically all orders are placed between X_0 and $2X_0$.

5.4 The desired execution probability and time horizon

We see that execution probability and time horizon are not independent. However, we show that the investor always finds a suitable limit price which has a positive probability of leading to a given execution probability and time horizon. For $p \geqslant X_0$ introduce the function $g_p : [0, \infty) \to [0, 1]$ with $g_p\left(t\right) = P\left(\sup_{0 \leqslant s \leqslant t} X_s \geqslant p\right)$ which assigns to each time t and limit price p the corresponding execution probability. Assume that the stock price $\left(X_t\right)_{t \geqslant 0}$ is given as in (2.2). Some properties of g_p are stated:

Lemma 5.21 *g_p is strictly increasing.*

$$\lim_{t\to\infty} g_p(t) = \begin{cases} 1 & \text{if } \mu \geqslant \frac{\sigma^2}{2} \\ \frac{X_0}{p} & \text{if } \mu < \frac{\sigma^2}{2} \end{cases}$$

and

$$\lim_{t\to 0} g_p(t) = \begin{cases} 0 & \text{if } p > X_0 \\ 1 & \text{if } p = X_0 \end{cases}.$$

Proof. It is clear that g_p is nondecreasing. g_p is strictly increasing, since the stock price process is not constant (a.s.). The second part is straightforward. ∎

Lemma 5.22 *Let $t \geqslant 0$. Then*

$$\lim_{p\to\infty} g_p(t) = 0$$

and

$$g_{X_0}(t) = 1.$$

Furthermore, $g_p(t)$ is continuous in p.

Proof. The first part is clear. The continuity follows from the continuity of the stock price process. ∎

Consider the problem of an investor, who wants to submit a limit order. Assume that he has to decide on his limit price. He knows that his time horizon is t and his desired execution probability is α. By the previous lemma, he always finds a limit price p such that $\alpha = g_p(t)$. This means that, by submitting his order at price p and waiting until time t, his execution probability is α. If we include cancellation here, we have

$$g_p^c(t) = P\left(\sup_{0\leqslant s\leqslant \min\{\tau^p,t\}} X_s \geqslant p\right).$$

Again, we get

Lemma 5.23 *For all $t \geqslant 0$*

$$\lim_{p\to\infty} g_p^c(t) = 0$$

and

$$g_{X_0}^c(t) = 1.$$

Therefore, again, the investor finds an optimal limit price p which gives him exactly the desired time horizon and execution probability. (Note that with cancellation we assume that the investor has time horizon t but still cancels the order at the random time τ^p. This τ^p can be due to unforeseen changes in the desired horizon of the investor.)

5.5 A short note on the interplay between limit price and waiting time

Assume the following situation: An investor decides at time 0 that he pays X_0 and gets p once the stock price reaches p. He has no other possibilities to trade or hedge. He is an impatient investor who has the following utility function

$$\begin{aligned} U &: \mathbb{R}_+^2 \to \mathbb{R} \\ U(p,t) &= U_1(p)\, e^{-rt} \end{aligned}$$

for some $r > 0$. We assume that U_1 is differentiable and strictly concave with $U_1(p) > 0$ for $p \geqslant 0$. He wants to maximize his expected utility, if the time horizon is $\mathcal{T}_H^p$:

$$U_1(p - X_0)\, E\left(e^{-r\mathcal{T}_H^p}\right).$$

We obtain:

Lemma 5.24 *Let $r > 0$. Assume that the stock price process is given as in (2.2) with $\nu > 0$. Then a necessary condition for the existence of a solution to*

$$\sup_{p \geqslant X_0} U_1(p - X_0)\, E\left(e^{-r\mathcal{T}_H^p}\right) \tag{5.2}$$

is

$$\frac{U_1'(p - X_0)}{U_1(p - X_0)} = \frac{\alpha}{p},$$

where $\alpha = \sqrt{\nu^2 + \frac{2r}{\sigma^2}} - \nu$. In the case $U_1(p) = p$, we obtain that a solution to (5.2) is given by $p = \frac{\alpha}{\alpha - 1} X_0$ with

$$\sup_{p \geqslant X_0} (p - X_0)\, E\left(e^{-r\mathcal{T}_H^p}\right) = \left(\frac{X_0}{\alpha - 1}\right)\left(\frac{X_0}{p}\right)^{\alpha}$$

if $\alpha > 1$. For $\alpha \leqslant 1$, there is no solution to (5.2), we would choose the limit price as high as possible.

Proof. We look at the Laplace transform of the time horizon

$$E\left(e^{-r\mathcal{T}_H^p}\right) = E\left(e^{-r\mathcal{T}_H^p}; \mathcal{T}_H^p < \infty\right)$$
$$= \begin{cases} \left(\frac{X_0}{p}\right)^{\sqrt{\nu^2 + \frac{2r}{\sigma^2}} - \nu} & \text{for} \quad X_0 \leqslant p \\ \left(\frac{p}{X_0}\right)^{\sqrt{\nu^2 + \frac{2r}{\sigma^2}} + \nu} & \text{for} \quad p \leqslant X_0 \end{cases}.$$

and only consider the case $p > X_0$. Then

$$E\left(e^{-r\mathcal{T}_H^p}\right) = \left(\frac{X_0}{p}\right)^{\sqrt{\nu^2 + \frac{2r}{\sigma^2}} - \nu}.$$

We consider

$$U_1(p-X_0)\cdot\left(\frac{X_0}{p}\right)^{\sqrt{\nu^2+\frac{2r}{\sigma^2}}-\nu}$$

and differentiate this with respect to p. Denote $\alpha=\sqrt{\nu^2+\frac{2r}{\sigma^2}}-\nu$. Then

$$\begin{aligned}
&\frac{d}{dp}U_1(p-X_0)\cdot\left(\frac{X_0}{p}\right)^{\alpha}\\
=\;& U_1'(p-X_0)\cdot\left(\frac{X_0}{p}\right)^{\alpha}-\alpha\frac{1}{X_0}U_1(p-X_0)\cdot\left(\frac{X_0}{p}\right)^{\alpha+1}=0\\
\Longleftrightarrow\;& \frac{U_1'(p-X_0)}{U_1(p-X_0)}=\frac{\alpha}{p}.
\end{aligned}$$

We look at the second derivative and obtain:

$$\begin{aligned}
&U_1''(p-X_0)\cdot\left(\frac{X_0}{p}\right)^{\alpha}-U_1'(p-X_0)\cdot\alpha X_0^{\alpha}\left(\frac{1}{p}\right)^{\alpha+1}\\
&-\alpha\frac{1}{X_0}U_1'(p-X_0)\cdot\left(\frac{X_0}{p}\right)^{\alpha+1}+\alpha(\alpha+1)\frac{1}{X_0}U_1(p-X_0)\cdot X_0^{\alpha+1}\left(\frac{1}{p}\right)^{\alpha+2}
\end{aligned}$$

If $U_1(p)=p$ and $\alpha\neq1$, we get

$$p=\frac{\alpha}{\alpha-1}X_0$$

as necessary condition and for the second derivative:

$$\begin{aligned}
&-2\alpha X_0^{\alpha}\left(\frac{1}{p}\right)^{\alpha+1}+\alpha(\alpha+1)(p-X_0)\cdot X_0^{\alpha}\left(\frac{1}{p}\right)^{\alpha+2}\\
=\;&\left(X_0^{\alpha}\left(\frac{1}{p}\right)^{\alpha+1}\right)\left(-2\alpha+\alpha(\alpha+1)(p-X_0)\cdot\frac{1}{p}\right)<0\\
\Longleftrightarrow\;&\left(-2\alpha+\alpha(\alpha+1)(p-X_0)\cdot\frac{1}{p}\right)<0\\
\Longleftrightarrow\;&\left(-2\alpha+\alpha(\alpha+1)-\alpha(\alpha+1)\cdot\frac{X_0}{p}\right)<0\\
\Longleftrightarrow\;&\alpha\left(-1+\alpha-(\alpha+1)\cdot\frac{X_0}{p}\right)<0.
\end{aligned}$$

Plug in the previously found necessary equation for the optimal p.

$$\begin{aligned}
&\alpha\left(-1+\alpha-(\alpha+1)\cdot\frac{X_0}{\frac{\alpha}{\alpha-1}X_0}\right)=\alpha\left(-1+\alpha-(\alpha+1)\cdot\frac{\alpha-1}{\alpha}\right)\\
=\;&\alpha\left(-1+\alpha-\frac{\alpha^2-1}{\alpha}\right)=-\alpha+\alpha^2-\alpha^2+1\\
=\;&1-\alpha<0\Longleftrightarrow\alpha>1.
\end{aligned}$$

Then

$$\sup_{p \geqslant X_0} E\left(U\left(p, T_H^p\right)\right) = \left(\frac{\alpha}{\alpha-1}X_0 - X_0\right)\left(\frac{X_0}{p}\right)^{\alpha} = \left(\frac{X_0}{\alpha-1}\right)\left(\frac{X_0}{p}\right)^{\alpha}.$$

For $\alpha = 1$, we obtain

$$(p - X_0)\, E\left(e^{-rT_H^p}\right) = (p - X_0) \cdot \left(\frac{X_0}{p}\right) = X_0 - \frac{X_0^2}{p}$$

This has no maximum. For $\alpha < 1$ we also do not get a solution. ■

6 An extension to a large trader model

The previous concepts relied crucially on the assumption that the investor's behaviour has no feedback effect on the transaction price $(X_t)_{t \geqslant 0}$. Here we want to relax this assumption in two steps. First, we consider a large trader who has to go "into the book" and show that under certain conditions, no arbitrage opportunities exist. In a second step, we also assume that submitting limit orders has an effect on the current (and future) transaction price. We will highlight several interesting and important aspects.

6.1 A first step towards the large trader model and no arbitrage

Here we show that there are no arbitrage opportunities for the large trader. This question has been solved in great generality, see [DS98], if we allow all traders to trade at the current stock price. Therefore, we consider here the trading activities of a large trader who has to go "into the book", meaning that he cannot sell or buy his stocks at the current transaction price but only at the currently prevailing limit prices (with their respective quantities). Here we allow for temporary and permanent price impacts, however permanent impacts are restricted to changes in the order book and do not affect the stock price process.

We define arbitrage opportunities and admissible trading strategies. This section uses the setup from Çetin, Jarrow and Protter([CJP04]), however we adapt it to our model. In particular, it is crucial for us that the stock price per share depends on the history of the trading strategy. We show that in our previously constructed model of the order book, no arbitrage opportunities arise if there exists a probability measure Q which is equivalent to P such that $(X_t)_{t \geqslant 0}$ is a local $Q-$martingale. The market is extended with a money market account.

Definition 6.1 (Trading strategy) *A trading strategy is a tuple* $\left((H_t, \beta_t)_{t\in[0,T]}\right)$ *where* H_t *represents the trader's stock holding at time t and* β_t *the trader's money market account position at time t. We impose the following conditions:*

(a) H_t *and* β_t *are predictable processes with* $H_{0-} = 0 = \beta_{0-}$,

(b) $H_T = 0$ *(at latest we have to liquidate our stock positions at time* T.*)*,

(c) $H_t \in \mathcal{Z}$, *since we only want to allow the number of shares traded to be integers.*

For the large trader, not the transaction price is relevant, but the distribution of limit orders in the book. Here we assume that the behaviour of the large trader does not affect the stock price. The requirements on the trading strategy are consistent with the previously constructed order book, if we assume that the trader has to go "into the book", when buying or selling shares.

Definition 6.2 (Stock price process and the price impact) *Here we look at the price per share which the large investor has to pay, if his aggregate holdings are $H_t \in \mathbb{Z}$ shares at time t and if his strategy is $(H_s)_{0\leqslant s<t-}$ up to time t and denote it by $X\left(t, H_t, (H_s)_{0\leqslant s<t}\right)$.*

1. We assume that $X\left(t, H_t, (H_s)_{0\leqslant s<t}\right)$ is $\mathcal{F}_t$–measurable and non-negative for all possible trading strategies $(H_s)_{0\leqslant s<t}$ and all $H_t \in \mathbb{Z}$.

2. $$H_t \to X\left(t, H_t, (H_s)_{0\leqslant s<t}\right)$$
   is a.e. non-decreasing in H_t, $P-$ a.s. for all possible trading strategies $(H_s)_{0\leqslant s<t}$ (i.e. $x \leqslant y$ implies $X\left(t, x, (H_s)_{0\leqslant s<t}\right) \leqslant X\left(t, y, (H_s)_{0\leqslant s<t}\right)$ P–a.s., a.e.).

3. $X\left(\cdot, H_\cdot, (H_s)_{0\leqslant s<\cdot}\right)$ is a (càdlàg) semimartingale for all possible trading strategies $(H_s)_{0\leqslant s\leqslant \cdot}$.

4. $X\left(t, 0, (H_s)_{0\leqslant s<t}\right) = X_t$ for all possible trading strategies $(H_s)_{0\leqslant s<t}$. (This reflects the assumption that there are no permanent price impacts on the stock price.)

Remark 6.3 $X\left(t, H_t, (H_s)_{0\leqslant s<t}\right)$ *can take on the value* 0 *or* $+\infty$ *(This happens if the order book is empty at time* t*.). With the notation used earlier, we can calculate* $X\left(t, H_t, (H_s)_{0\leqslant s<t}\right)$. *As an example, we compute* $X(t, H_t, 0)$. *Assume there exists* $n \in \mathbb{N}$ *such that there are* H_t *orders in the book up to price* $a_t^{(n)}$, *i.e.*

$$\sum_{i=1}^{n} O^s\left(\{t\}, \left(a_t^{(i-1)}, a_t^{(i)}\right], [0,\infty), ds\right) = H_t.$$

To buy H_t *shares, we have to pay*

$$X(t, H_t, 0) = \sum_{i=1}^{n} a_t^{(i)} \cdot O^s\left(\{t\}, \left(a_t^{(i-1)}, a_t^{(i)}\right], [0,\infty)\right),$$

where $a_t^{(0)}$ *denotes the current stock price* X_t, $a_t^{(1)}$ *the best ask price,* $a_t^{(2)}$ *the second best ask price etc.*

We want to point out here, that the assumptions in the previous definition can easily be imposed and incorporated in our model of the order book. Therefore, standard assumptions which are made in the large trader theory, and basically come from conditions such that certain results in mathematical finance hold or can be proved, can now be based on a reasonable framework. Assumption 1 says that the price per share which the large trader has to pay is nonnegative and F_t–measurable for any possible trading strategy. In our model, the large trader can only "pick up" orders which are in the book. Clearly, the nonnegativity of the price is satisfied and if we take as F_t the augmented filtration which is generated by the stock price, the order arrival and the cancellation process, the F_t−measurability is obvious. As for Assumption 2, if the large trader wants to buy more shares, the price per share is nondecreasing. This is satisfied in our order book model, since the trader has to go into the book and does not e.g. get a discount if he buys a large quantity. The third assumption needs more attention. Of course, if we allow arbitrary trading strategies, then this might not be satisfied. However, we can only allow such strategies which lead to a semimartingale for the stock price per share (Any finite trading strategy which does not try to buy stocks when the order book is empty would be one example for this.). Finally the last assumption sets the price per share which one has to pay if one does not want to trade equal to the current stock price. Of course, this just amounts to defining a hypothetical price if no one trades. This feature is a good interpretation of our exogenously given transaction price which we introduced in our order book model. In other words, we would just define the transaction price as the hypothetical price if no one buys or sells shares to make our model compatible with the above assumptions.

To talk about arbitrage opportunities we first need to discuss self-financing strategies.

Definition 6.4 (Self-financing strategy) *We define a self-financing trading strategy as a trading strategy* $\left((H_t, \beta_t)_{t\in[0,T]}\right)$ *where*

(a) H_t *is càdlàg with finite quadratic variation* $([H,H]_t < \infty)$,

(b) $\beta_0 = -H_0 X_0$ *and*

(c) *for* $0 < t \leqslant T$,

$$\begin{aligned}
\beta_t &= \beta_0 + H_0 X_0 + \int_0^t H_{u-}\, dX_u - H_t X_t \\
&\quad - \sum_{0\leqslant u\leqslant t} \Delta H_u \left(X\left(u, \Delta H_u, (H_s)_{0\leqslant s<u}\right) - X_u\right) \\
&= \int_0^t H_{u-}\, dX_u - H_t X_t - \sum_{0\leqslant u\leqslant t} \Delta H_u \left(X\left(u, \Delta H_u, (H_s)_{0\leqslant s<u}\right) - X_u\right)
\end{aligned}$$

Remark 6.5 *This definition can be justified as follows: Let t be a fixed time, $(H_s)_{s\in[0,t]}$ a trading strategy and let (σ_n) be a sequence of random partitions of $[0,t]$ tending to identity in the following form*

$$\sigma_n : 0 = T_0^n \leqslant T_1^n \leqslant \cdots \leqslant T_{k_n}^n = t,$$

where the T_k^n's are stopping times. We define β_t as the limit

$$\beta_t = \beta_0 - \lim_{n\to\infty} \sum_{k\geqslant 1} \left(H_{T_k^n} - H_{T_{k-1}^n}\right) \left(X\left(T_k^n, H_{T_k^n} - H_{T_{k-1}^n}, \left(H_{T_l^n}\right)_{0\leqslant l\leqslant k-1}\right)\right)$$

in ucp, whenever it exists. We compute

$$\begin{aligned}\beta_t &= \beta_0 - \lim_{n\to\infty}\sum_{k\geqslant 1}\left(H_{T_k^n} - H_{T_{k-1}^n}\right)\left(X\left(T_k^n, H_{T_k^n} - H_{T_{k-1}^n}, \left(H_{T_l^n}\right)_{0\leqslant l\leqslant k-1}\right)\right)\\ &= \int_0^t H_{u-}\, dX_u - H_t X_t\\ &\quad - \lim_{n\to\infty}\sum_{k\geqslant 1}\left(H_{T_k^n} - H_{T_{k-1}^n}\right)\left(X\left(T_k^n, H_{T_k^n} - H_{T_{k-1}^n}, \left(H_{T_l^n}\right)_{0\leqslant l\leqslant k-1}\right) - X\left(T_k^n, 0, 0\right)\right)\end{aligned}$$

From this, we obtain

$$\begin{aligned}\beta_t &= \int_0^t H_{u-}\, dX_u - H_t X_t\\ &\quad - \sum_{0\leqslant u\leqslant t}\Delta H_u\left(X\left(u, \Delta H_u, (H_s)_{0\leqslant s<u}\right) - X_u\right).\end{aligned}$$

A similar calculation can be found in [CJP04], therefore we omit the details.

In this setup, several possibilities for defining the value of a portfolio occur. We are mainly concerned with the marked-to-market value and the liquidity cost.

Definition 6.6 (marked-to-market value) *We define the marked-to-market value of a trading strategy as the classical value given by*

$$\beta_t + H_t X_t.$$

Definition 6.7 (liquidity cost) *The liquidity cost of a self-financing trading strategy is defined as*

$$\begin{aligned}L_t &= \int_0^t H_{u-}\, dX_u - H_t X_t - \beta_t\\ &= \sum_{0\leqslant u\leqslant t}\Delta H_u\left(X\left(u, \Delta H_u, (H_s)_{0\leqslant s<u}\right) - X_u\right).\end{aligned}$$

This leads to

$$\beta_t + H_t X_t = \int_0^t H_{u-}\, dX_u - L_t.$$

We immediately see that the liquidity costs are nonnegative and nondecreasing in t.

Now we have built the foundation for discussing arbitrage opportunities.

Definition 6.8 (arbitrage opportunity) *An arbitrage opportunity is a self-financing trading strategy (H,β) such that $P(\beta_T \geqslant 0) = 1$ and $P(\beta_T > 0) > 0$.*

Definition 6.9 ($\alpha-$admissible) *Let $\alpha \geqslant 0$. A self-financing trading strategy (H,β) is said to be $\alpha-$admissible if $(H,\beta)\in\Theta_\alpha$ where*

$$\Theta_\alpha = \left\{(H,\beta)\;\middle|\; (H,\beta)\ \textit{s.f.t.s.},\ \int_0^t H_{u-}\, dX_u \geqslant -\alpha \quad P-\text{a.s.},\ \forall t\in[0,T]\right\}.$$

This leads to the following lemma.

Lemma 6.10 *Assume there exists a probability measure Q which is equivalent to P such that $(X_t)_{t\geqslant 0}$ is a local $Q-$martingale. If $(H,\beta) \in \Theta_\alpha$ for some α, then $\beta_t + H_t X_t$ is a $Q-$supermartingale.*

Proof. If $(X_t)_{t\geqslant 0}$ is a local $Q-$martingale, then $\int_0^t H_{s-}\, dX_s$ is a local $Q-$martingale. Since the trading strategy is $\alpha-$admissible, it is also a supermartingale. Furthermore

$$\beta_t + H_t X_t = \int_0^t H_{u-}\, dX_u - L_t$$

with L_t non-negative and non-decreasing. Therefore the claim follows. ■

From this, we obtain a sufficient condition for no-arbitrage.

Lemma 6.11 *Assume there exists a probability measure Q which is equivalent to P such that $(X_t)_{t\geqslant 0}$ is a local $Q-$martingale. This implies that there is no arbitrage for $(H,\beta) \in \Theta_\alpha$ for any α.*

Proof. Let $(H,\beta) \in \Theta_\alpha$ be an admissible self-financing trading strategy for some α. We already know that $\beta_t + H_t X_t$ is a $Q-$supermartingale. Together with

$$E_Q\left(\beta_0 + H_0 X_0\right) = 0$$

this leads to

$$E_Q\left(\beta_T\right) = E_Q\left(\beta_T + H_T X_T\right) \leqslant 0$$

However, by the definition of an arbitrage opportunity, we would need

$$E_Q\left(\beta_T\right) > 0.$$

■

Remark 6.12 *The converse is in general not true, even if we go over to no free lunch with vanishing risk. Observe the following: We allowed the price per share which the large investor has to pay to depend on the history of transactions by the large trader, which is a crucial feature of our model.*

6.2 A second step towards the large trader model and no arbitrage

Here we consider an extension of the previous setup. Now we have a "real" large trader. He can either submit limit orders or market orders. If he submits market orders, he has to go "into the book" as in the previous subsection. This feature would lead, under mild conditions, to no arbitrage. However, if the large trader has in addition the possibility to submit limit orders, this has also an effect on the transaction price. Therefore the large trader has market manipulating abilities. This might offset the losses which he has to suffer. We give conditions under which this model still admits no arbitrage. Usually, one differentiates temporary and permanent price impacts. Here the situation is as follows: Limit orders have a (random) permanent (they influence the stock price) and a temporary impact (This is due to the fact that the stock price process can have jumps and therefore it might happen that the price jumps over a limit sell order (or below for buy orders).). Market orders have a temporary impact (the trader has to go into the book). They also have a permanent impact on the state of the order book, because market orders consume liquidity, i.e. limit orders.

We look separately at the number of buy limit orders and sell limit orders which the large trader has submitted. Define the total number of submitted buy and limit orders by time t by $\mathcal{A}_t$ and $\mathcal{B}_t$ respectively.

For the moment we allow only the following strategies for the large trader. The large trader can

1. submit market buy or sell orders, which would be executed by going "into the book" (the part which cannot be executed due to an empty order book is cancelled immediately),
2. submit buy or sell limit orders, which lead to an immediate permanent price impact and would be executed once they are hit by the stock price,
3. cancel any limit orders which he previously submitted and which are still in the book. This would lead to an immediate permanent price impact.
4. submit orders only at finitely many times, be it either market or limit orders.
5. Two different kinds of orders cannot be submitted at the same time.

Definition 6.13 (Trading strategy and order book strategy of the large trader) *Fix dates $0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T$. A* ***trading strategy of the large trader*** *consists of the tuple $((\mathcal{A}_t), (\mathcal{B}_t), (H_t), (\beta_t))_{t\in[0,T]}$ where $\mathcal{A}_t$ denotes the total number of sell limit orders which he has submitted by time t, $\mathcal{B}_t$ the total number of buy limit orders, H_t the number of orders which he has acquired through market orders at time t and β_t represents the trader's money market account position at time t. We also assume that he cannot submit two different types of orders or limit orders with different limit prices at the same time. Furthermore, we impose the following conditions: $\mathcal{A}_t, \mathcal{B}_t, H_t$ and β_t are predictable processes starting at 0 at time $0-$. $\mathcal{A}_t, \mathcal{B}_t \in \mathbb{N}_0$ since we only want to allow integers for the number of limit orders in the book. Furthermore, H_t, $\mathcal{A}_t$ and $\mathcal{B}_t$ are càdlàg with finite quadratic variation and can only jump at the times t_i, $1 \leqslant i \leqslant n$. We use the term* ***order book strategy*** *for $((\mathcal{A}_t), (\mathcal{B}_t))_{t\in[0,T]}$.*

With the assumption that no two different types of orders can be submitted at the same time, we basically want to exclude that we can have two different stock prices at the same time for the same $\omega \in \Omega$. In practice, this makes sense, since, even though we can usually cancel any orders for free, there is at least some small time interval during which we are exposed to the possibility that our limit order is executed or the order book changes.

At the moment we only allow a discrete strategy for the large trader, i.e. he can participate in the market only at finitely many times.

We define the stock price process which results from the permanent price impact of submitting limit orders.

Definition 6.14 (Stock price process) *Assume that up to time t, $\mathcal{A}_t \in \mathbb{R}_+$ buy limit orders and $\mathcal{B}_t \in \mathbb{R}_+$ sell limit orders were submitted. Then the stock price process follows*

$$X_t \cdot g_t(\mathcal{A}_t, \mathcal{B}_t),$$

where X_t and $g_t : \Omega \times \mathbb{R}_+^2 \to \mathbb{R}$ are such that

1. *$X_t \cdot g_t(\mathcal{A}_t, \mathcal{B}_t)$ is $\mathcal{F}_t$−measurable and non-negative for all possible order book strategies of the large trader $(\mathcal{A}_t, \mathcal{B}_t)_{0 \leqslant t \leqslant T}$.*
2. *X_t and g_t are independent.*
3. *$X_t \cdot g_t(\mathcal{A}_t, \mathcal{B}_t)$ is a càdlàg semimartingale for all possible order book strategies of the large trader $(\mathcal{A}_t, \mathcal{B}_t)_{0 \leqslant t \leqslant T}$.*
4. *$X_t \cdot g_t(0,0) = X_t$, $t \in [0,T]$.*

Definition 6.15 (Self-financing strategy) *We define a self-financing trading strategy as a trading strategy of the large trader $((\mathcal{A}_t), (\mathcal{B}_t), (H_t), (\beta_t))_{t \in [0,T]}$ where*

(a) *$\beta_0 = -H_0 X_0 g_0(\mathcal{A}_0, \mathcal{B}_0)$ and*

(b) *for $0 < t \leqslant T$,*

$$\beta_t = \underbrace{\int_0^t H_{u-}\, dX_u g_u(\mathcal{A}_u, \mathcal{B}_u)}_{\text{gains/losses from market orders}} - \bar{H}_t X_t g_t(\mathcal{A}_t, \mathcal{B}_t) + G_t - L_t, \tag{6.1}$$

where G_t denotes the gains/ losses from limit orders up to time t, $\bar{H}_t$ the total number of orders which are in the portfolio at time t (including those acquired through market orders and limit orders) and L_t the losses from the temporary price impact caused by market orders (The latter was investigated more thoroughly in the previous section. We only need that $L_t \geqslant 0$.). We will look at G_t in more detail at a later stage.

We still need to define admissible strategies.

Definition 6.16 (α−admissible) *Let $\alpha \geqslant 0$. A self-financing trading strategy $((\mathcal{A}_t), (\mathcal{B}_t), (H_t), (\beta_t))_{t \in [0,T]}$ is said to be α−admissible if $((\mathcal{A}_t), (\mathcal{B}_t), (H_t), (\beta_t))_{t \in [0,T]} \in \Theta_\alpha$ where*

$$\begin{aligned}\Theta_\alpha \;=\; & \{(\mathcal{A}, \mathcal{B}, H, \beta) \mid (\mathcal{A}, \mathcal{B}, H, \beta) \text{ s.f.t.s.}, \\ & \left. \int_0^t H_{u-}\, dX_u g_u(\mathcal{A}_u, \mathcal{B}_u) + G_t \geqslant -\alpha \quad P-\text{a.s.},\ t \in [0,T] \right\}.\end{aligned}$$

Definition 6.17 (arbitrage opportunity) *For an arbitrage opportunity we can of course only look at the value of stocks after liquidation due to the possible liquidation losses. An arbitrage opportunity is a self-financing trading strategy $((\mathcal{A}_t), (\mathcal{B}_t), (H_t), (\beta_t))_{t \in [0,T]}$ such that $\mathcal{A}_T = \mathcal{B}_T = H_T = 0$ and $P(\beta_T \geqslant 0) = 1$ with $P(\beta_T > 0) > 0$.*

Given a strategy $((\mathcal{A}_t), (\mathcal{B}_t), (H_t), (\beta_t))_{t \in [0,T]}$ and a stock price process $(X_t)_{t \in [0,T]}$ denote the number of shares the large trader has at time t by $\bar{H}_t$ (This is composed of all market orders H_t and the limit orders which have been executed.). Then the portfolio wealth (book value) at time t is given by

$$\beta_t + \bar{H}_t \cdot X_t g_t(\mathcal{A}_t, \mathcal{B}_t).$$

We split the gains into two components: The gains or losses caused by order submission and the gains or losses caused by stock price changes. Set $\pi_t := \bar{H}_t - \bar{H}_{t-}$. Assume that at time t_i, the market maker obtains π_{t_i} shares for the price $X_{t_i}g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i})$ through a limit or market order. Then the (paper) gains of this transaction at time $t \geqslant t_i$ are given by

$$\begin{aligned} & \pi_{t_i} \cdot X_t g_t(\mathcal{A}_t,\mathcal{B}_t) - \pi_{t_i} \cdot X_{t_i} g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i}) \\ = & \pi_{t_i} \cdot X_t g_t(\mathcal{A}_t,\mathcal{B}_t) - \pi_{t_i} \cdot X_t g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i}) + \pi_{t_i} \cdot X_t g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i}) - \pi_{t_i} \cdot X_{t_i} g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i}) \\ = & \pi_{t_i} \cdot X_t \left(g_t(\mathcal{A}_t,\mathcal{B}_t) - g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i})\right) + \pi_{t_i} \cdot g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i})(X_t - X_{t_i}) \end{aligned}$$

The first term describes the gains/ losses which are caused through changes in the influence of the order book on the stock price, the second term the gains/ losses which are caused by the change in the stock price.

If at time t_i, the stock price is at $X_{t_i}g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i})$ and the large trader submits a market order, his real gains are bounded from above by the previous expression. Indeed including the temporary price impact for market orders, which we denote by $L_t(\pi_t)$ at time t, the gains/ losses are

$$\pi_{t_i} \cdot X_t \left(g_t(\mathcal{A}_t,\mathcal{B}_t) - g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i})\right) + \pi_{t_i} \cdot g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i})(X_t - X_{t_i}) - L_{t_i}(\pi_{t_i}).$$

Lemma 6.18 *Under the above assumptions, in particular that we cannot submit different types of orders at the same time, the temporary impact $L_t(\pi_t)$ is nonnegative.*

Proof. This follows immediately from our setup of the order book since the large trader has to go "into the book". On the other hand, if he could first submit market orders and then at the same time, limit orders, it could happen that the temporary impact is negative. ∎

Remark 6.19 *The term*

$$\int_0^t H_{u-}\, dX_u g_u(\mathcal{A}_u,\mathcal{B}_u) + G_t$$

is bounded from above by

$$\int_0^t \bar{H}_{u-}\, dX_u g_u(\mathcal{A}_u,\mathcal{B}_u).$$

At which price are limit orders traded? Given a strategy $((\mathcal{A}_t),(\mathcal{B}_t),(H_t),(\beta_t))_{t\in[0,T]}$, the process $(X_t g_t(\mathcal{A}_t,\mathcal{B}_t))_{t\geqslant 0}$ is càdlàg. We consider three cases. First, if at time t the process $X_t g_t(\mathcal{A}_t,\mathcal{B}_t)$ is continuous, the price at which a limit order is executed (if it is executed) is equal to $X_t g_t(\mathcal{A}_t,\mathcal{B}_t)$. If we have a jump at time t and a sell limit order with limit price $X_{t-}g_{t-}(\mathcal{A}_{t-},\mathcal{B}_{t-})$, the sell order will not be executed if the jump is downward. It will be executed at a price at most $X_t g_t(\mathcal{A}_t,\mathcal{B}_t)$ if the jump is upward. In summary, we obtain that the maximum sell price of a limit order sold at time t is always $X_t g_t(\mathcal{A}_t,\mathcal{B}_t)$, similarly, the minimum price at which a buy limit order is executed is equal to $X_t g_t(\mathcal{A}_t,\mathcal{B}_t)$.

If we add up the gains for all market and limit orders, we obtain at most

$$\sum_{t_i\leqslant t}\left[\pi_{t_i} \cdot X_t g_t(\mathcal{A}_t,\mathcal{B}_t) - \pi_{t_i} \cdot X_{t_i} g_{t_i}(\mathcal{A}_{t_i},\mathcal{B}_{t_i})\right] - \sum_{\substack{t_i\leqslant t \\ \text{Order is a} \\ \text{market order}}} L_{t_i}(\pi_{t_i}).$$

Theorem 6.20 *Under the above assumptions, if in addition for every order book strategy $((\mathcal{A}_t),(\mathcal{B}_t))_{t\in[0,T]}$ there exists an equivalent local martingale measure Q under which $(X_t g_t(\mathcal{A}_t,\mathcal{B}_t))_{t\in[0,T]}$ is a local martingale, no arbitrage opportunities for the large trader exist.*

Proof. Start with a given self-financing strategy $((\mathcal{A}_t), (\mathcal{B}_t), (H_t), (\beta_t))_{t\in[0,T]}$. The gains process is bounded from above by

$$\sum_{t_i \leqslant t} \left[\pi_{t_i} \cdot X_t g_t(\mathcal{A}_t, \mathcal{B}_t) - \pi_{t_i} \cdot X_{t_i} g_{t_i}(\mathcal{A}_{t_i}, \mathcal{B}_{t_i})\right] - \sum_{\substack{t_i \leqslant t \\ \text{Order is a} \\ \text{market order}}} L_{t_i}(\pi_{t_i}).$$

Under the measure Q, we obtain

$$\begin{aligned} & E_Q\left(\sum_{t_i \leqslant t} \left[\pi_{t_i} \cdot X_t g_t(\mathcal{A}_t, \mathcal{B}_t) - \pi_{t_i} \cdot X_{t_i} g_{t_i}(\mathcal{A}_{t_i}, \mathcal{B}_{t_i})\right] - \sum_{\substack{t_i \leqslant t \\ \text{Order is a} \\ \text{market order}}} L_{t_i}(\pi_{t_i})\right) \\ \leqslant\ & E_Q\left(\sum_{t_i \leqslant t} \left[\pi_{t_i} \cdot X_t g_t(\mathcal{A}_t, \mathcal{B}_t) - \pi_{t_i} \cdot X_{t_i} g_{t_i}(\mathcal{A}_{t_i}, \mathcal{B}_{t_i})\right]\right) = 0. \end{aligned}$$

■

Remark 6.21 *Why should limit orders affect the current stock price? Consider a sell limit order. When the large trader submits it, this implies that the supply has increased. Furthermore, other traders might see that a probably informed trader wants to sell shares. Therefore they want to jump ahead of him and the stock price falls. On the other hand, the stock price could also increase if the supply in the book is low, i.e. there are just a few limit orders and there is another trader, call him trader 2, who would like to buy a large quantity of the stock. If he does so immediately, he would suffer quite a large price impact. But once the large trader submitted his limit order, there is enough liquidity and trader 2 could buy the stock (probably by deleting existing limit orders and submitting market orders or just by submitting market orders). All those effects (and many more) are observed in real stock markets.*

Remark 6.22 *Assume that the large trader is allowed to place limit orders at time t and cancel them immediately. If after submission and cancellation, the stock price process is at the same level again, we would have immediate arbitrage opportunities if*

$$\inf_{a,b\in\mathbb{R}_+} X_t \cdot g_t(a,b) \leqslant b_t^L$$

and

$$\sup_{a,b\in\mathbb{R}_+} X_t \cdot g_t(a,b) \geqslant a_t^L$$

where a_t^L is the minimum price of all buy limit orders which are in the book and were submitted by the large trader. Similarly, b_t^L is the maximum of all limit buy orders of the large trader. In other words, by just submitting limit orders, the trader would be able to execute one of his own limit orders.

6.3 Lower bounds on manipulable contingent claims.

Since the trader can influence the stock price by submitting or cancelling limit orders, the natural question of pricing manipulable contingent claims arises. This means, we consider contingent claims whose payoff may depend on the order book strategy of the large trader. We work with the setup from the previous subsection and extend the usual notion of contingent claims to the following:

Definition 6.23 (Manipulable contingent claim) *A* ***manipulable contingent claim*** *is a $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^2 \times [0,T])$ measurable mapping which is bounded from below. The holder is entitled to a contingent payment of $C\left(\omega, (\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,T]}\right)$ at time T, where $(\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,T]}$ denotes the order book strategy of the large trader.*

Examples would include the European call option with strike K, where we would choose $C\left(\omega, (\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,T]}\right) = \left(X_T g_T\left((\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,T]}\right) - K\right)^+$, the Asian option with $C\left(\omega, (\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,T]}\right) = \frac{1}{T}\int_0^T \max\left\{\left(K - X_s g_s\left((\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,s]}\right)\right), 0\right\} ds$ or the lookback option with $C\left(\omega, (\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,T]}\right) = \max\left\{\left(K - \max_{0\leqslant s\leqslant T} X_s g_s\left((\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,s]}\right)\right), 0\right\}$.

Here we look at the task of superreplicating those contingent claims. However, we first define the book value.

Definition 6.24 (Book value) *If the trader has $\bar{H}_t$ shares and β_t units of money in the bank account, the book value of his portfolio is defined as*

$$V_t^{(\bar{H}_t,\beta_t)} = \beta_t + \bar{H}_t \cdot X_t g_t(\mathcal{A}_t, \mathcal{B}_t).$$

We want to define superreplication prices in terms of book value. This has the advantage that we can abstract from losses which would arise if we had to deliver, say a certain number of shares, physically.

The next definition gives us the precise meaning of the superreplication price.

Definition 6.25 (Superreplication price) *The* ***superreplication price*** *$\Pi(C)$ of a contingent claim C is the infimum of all initial capitals v for which there exists an admissible trading strategy of the large trader $((\mathcal{A}_t), (\mathcal{B}_t), (H_t), (\beta_t))_{t\in[0,T]}$ such that at time T we have $V_T^{(\bar{H}_T,\beta_T)} \geqslant C\left((\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,T]}\right)$ $P-$ almost surely:*

$$\begin{aligned}\Pi(C) &= \inf\Big\{v \;\Big|\; \exists\left((\mathcal{A}_t), (\mathcal{B}_t), (H_t), (\beta_t)\right)_{t\in[0,T]} \text{ admissible} \\ &\qquad \text{with } V_T^{(\bar{H}_T,\beta_T)} \geqslant C\left((\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,T]}\right) \ P-\text{a.s.},\ V_{0-}^{(\bar{H}_{0-},\beta_{0-})} = v\Big\}.\end{aligned}$$

We can give a lower bound on $\Pi(C)$. Start with an initial capital v and an admissible large trader strategy $((\mathcal{A}_t), (\mathcal{B}_t), (H_t), (\beta_t))_{t\in[0,T]}$ such that $V_{0-}^{(\bar{H}_{0-},\beta_{0-})} = v$ and $V_T^{(\bar{H}_T,\beta_T)} \geqslant C\left((\mathcal{A}_t, \mathcal{B}_t)_{t\in[0,T]}\right)$ $P-$almost surely. For the book value $V_t^{(\bar{H}_t,\beta_t)}$ we can equivalently write

$$V_t^{(\bar{H}_t,\beta_t)} = v + \int_0^t H_{u-}\, dX_u g_u(\mathcal{A}_u, \mathcal{B}_u) + G_t - L_t$$

(with the notation from (6.1)). The assumptions were made such that $V_t^{(\bar{H}_t,\beta_t)}$ is a supermartingale for all measures $Q \in \mathcal{Q}$. With $V_0^{(\bar{H}_0,\beta_0)} = v$, we obtain

$$v \geqslant \sup_{Q\in\mathcal{Q}} E_Q\left[V_T^{(\bar{H}_T,\beta_T)}\right] \geqslant \sup_{Q\in\mathcal{Q}} E_Q\left[C\left((\mathcal{A}_t,\mathcal{B}_t)_{t\in[0,T]}\right)\right].$$

Then

$$\Pi(C) \geqslant \inf_{LTS} \sup_{Q\in\mathcal{Q}} E_Q\left[C\left((\mathcal{A}_t,\mathcal{B}_t)_{t\in[0,T]}\right)\right]$$

where LTS denotes the set of initial capitals v for which there exists at least one admissible large trader strategy with initial capital v that superreplicates the contingent claim C.

Remark 6.26 *In our general setup, it is not possible to give upper bounds. One reason is that there might be no sell order in the book at a time when we need to buy the stock. If one would want to find upper bounds one should make severe restrictions on either the contingent claims available or the number of orders which are in the book at any given point in time.*

6.4 How to sell one stock using limit orders

The task is as follows: At time 0, we submit a sell limit order with limit p. If this order has not been executed by time T, we use a market order to sell it.

Now we want to make the dependence on the limit order which the large trader submits explicit. Therefore, denote the transaction price at time t by X_t^p, $p \in \mathbb{R}_+ \cup \{\infty\}$, if the large trader submitted one limit order (with size one) at time 0. ($p \in \{0,\infty\}$ means that he did not submit a limit order).

Assume as before that we have only a permanent impact at the time of submission of a limit order. We assume that we can decompose the stock price as follows

$$X_t^p = c(p) \cdot X_t$$

where $c : \mathbb{R}_+ \cup \{\infty\} \to \mathbb{R}_+$ is a measure of the permanent impact. If there is no permanent impact, we have $c(p) = 1$. Assume that X_0^p is differentiable as a function of p.

Recall that the elasticity of X_0^p with respect to p is defined as

$$\epsilon_{X_0^p,p} = \frac{p}{X_0^p}\frac{dX_0^p}{dp}.$$

Our goal is to liquidate a stock position. We only want to use limit orders. When submitting a limit order, a transaction occurs only if the limit price is attained. This advantage, the absence of price risk, does not come without a cost. We cannot guarantee execution and the time to execution is random and depends on many factors such as the limit price, the state of the order book, the size, market conditions and further public and private information. For some investors, call them impatient, the opportunity cost of waiting can be significant whereas for others, call them patient, execution time is not such a critical factor. If immediate execution is needed, the market order or a limit order very close to the current price, is the right instrument to use. However, market orders are subject to significant liquidity risk, in particular for large orders in volatile markets and markets with a thin order book. In practice, investors usually submit both market and limit orders. This helps them to balance the risk of delaying execution against the risk of immediate execution.

Note that, if the limit order is not executed by time t, we have to use a market order to sell the stock. Denote the expected liquidation proceeds from using a market order at time t by $L(t,p)$. (Based on the setup from the previous sections, one could explicitly calculate $L(t,p)$.). Assume that $L(t,p)$ is nondecreasing in p.

We obtain the first result:

Lemma 6.27 *Assume that the stock price $(X_t)_{t\geqslant 0}$ is given as in (2.2) with $\mu = 0$. If $\epsilon_{X_0^p,p} \geqslant 1$, i.e. X_0^p is elastic with respect to p, it is better to submit an even higher limit order when liquidating one stock with a limit order submitted at time 0.*

Proof. The expected liquidation proceeds at time T are

$$p \cdot P\left(X_t^p \geqslant p \text{ for some } t \in [0,T]\right) + L(T,p).$$

We obtain

$$\begin{aligned} & p \cdot P\left(X_t^p \geqslant p \text{ for some } t \in [0,T]\right) \\ = \; & p\left(1-\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)+p^{\frac{2\mu}{\sigma^2}}\frac{\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)}{\exp\left\{\frac{2\ln\left(X_0^p\right)\mu}{\sigma^2}\right\}}\right). \end{aligned}$$

Assume that $\mu = 0$. Then

$$\begin{aligned} & p\left(1-\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)+p^{\frac{2\mu}{\sigma^2}}\frac{\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)}{\exp\left\{\frac{2\ln\left(X_0^p\right)\mu}{\sigma^2}\right\}}\right) \\ = \; & p\left(1-\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)}{\sigma\sqrt{T}}\right)+\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)}{\sigma\sqrt{T}}\right)\right) = p\left(2-2\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)}{\sigma\sqrt{T}}\right)\right). \end{aligned}$$

As long as this term is increasing, it is in any case better to submit a higher limit order p. We calculate

$$\begin{aligned} & \frac{d}{dp}\left(p\left(2-2\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)}{\sigma\sqrt{T}}\right)\right)\right) \\ = \; & 2-2\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)}{\sigma\sqrt{T}}\right)+p\left(-2f\left(\frac{\ln\left(\frac{p}{X_0^p}\right)}{\sigma\sqrt{T}}\right)\cdot\frac{1}{\sigma\sqrt{T}}\left(\frac{1}{p}-\frac{1}{X_0^p}\frac{dX_0^p}{dp}\right)\right), \end{aligned}$$

where $f(x)$ denotes the density of the standard normal distribution. This is an explicit equation for X_0^p and it is clearly positive if

$$p\left(-2f\left(\frac{\ln\left(\frac{p}{X_0^p}\right)}{\sigma\sqrt{T}}\right)\cdot\frac{1}{\sigma\sqrt{T}}\left(\frac{1}{p}-\frac{1}{X_0^p}\frac{dX_0^p}{dp}\right)\right)\geqslant 0.$$

This holds iff

$$\begin{aligned}\frac{1}{p}-\frac{1}{X_0^p}\frac{dX_0^p}{dp} &\leqslant 0\\ &\Longleftrightarrow \frac{1}{p}\leqslant\frac{1}{X_0^p}\frac{dX_0^p}{dp}\\ &\Longleftrightarrow 1\leqslant\frac{p}{X_0^p}\frac{dX_0^p}{dp}.\end{aligned}$$

Therefore, if the elasticity of X_0^p with respect to p is greater or equal to 1, then it is better to submit an even higher limit order. ■

Precise details for the case $\mu\neq 0$ are as follows: Assume that the elasticity of X_0^p with respect to p is greater or equal to 1 and

$$1-\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)+\left(\frac{p}{X_0^p}\right)^{\frac{2\mu}{\sigma^2}}\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)\left(1+\frac{2\mu}{\sigma^2}-\frac{2\mu}{\sigma^2}\frac{p}{X_0^p}\frac{dX_0^p}{dp}\right)\geqslant 0$$

Then we should use a higher limit price. As before we have

$$\begin{aligned}&p\cdot P\left(X_t^p\geqslant p \text{ for some } t\in[0,T]\right)\\ =\;& p\left(1-\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)+p^{\frac{2\mu}{\sigma^2}}\frac{\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)}{\exp\left\{\frac{2\ln\left(X_0^p\right)\mu}{\sigma^2}\right\}}\right)\end{aligned}$$

and now differentiate with respect to p. After some calculations, the derivative yields

$$\begin{aligned}&1-\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)+p^{\frac{2\mu}{\sigma^2}}\frac{\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)}{\exp\left\{\frac{2\ln\left(X_0^p\right)\mu}{\sigma^2}\right\}}\\ &-f\left(\frac{\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)\cdot\frac{1}{\sigma\sqrt{T}}\left(1-\frac{p}{X_0^p}\cdot\frac{dX_0^p}{dp}\right)+\frac{2\mu}{\sigma^2}p^{\frac{2\mu}{\sigma^2}}\frac{\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)}{\exp\left\{\frac{2\ln\left(X_0^p\right)\mu}{\sigma^2}\right\}}\\ &+p^{\frac{2\mu}{\sigma^2}}\frac{f\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)\cdot\frac{1}{\sigma\sqrt{T}}\left(-1+\frac{p}{X_0^p}\cdot\frac{dX_0^p}{dp}\right)-\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)\cdot\frac{2\mu}{\sigma^2}\frac{p}{X_0^p}\frac{dX_0^p}{dp}}{\exp\left\{\frac{2\ln\left(X_0^p\right)\mu}{\sigma^2}\right\}}\end{aligned}$$

Further estimates and using the assumptions lead to

$$\begin{aligned}
&\frac{d}{dp}\left(p \cdot P\left(X_t^p \geqslant p \text{ for some } t \in [0,T]\right)\right)\\
= \;& 1-\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)+\left(\frac{p}{X_0^p}\right)^{\frac{2\mu}{\sigma^2}}\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)\left(1+\frac{2\mu}{\sigma^2}-\frac{2\mu}{\sigma^2}\frac{p}{X_0^p}\frac{dX_0^p}{dp}\right)\\
&-f\left(\frac{\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)\cdot\frac{1}{\sigma\sqrt{T}}\left(1-\frac{p}{X_0^p}\cdot\frac{dX_0^p}{dp}\right)\\
&+f\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)\cdot\frac{1}{\sigma\sqrt{T}}\left(\left(\frac{p}{X_0^p}\right)^{\frac{2\mu}{\sigma^2}}\left(-1+\frac{p}{X_0^p}\cdot\frac{dX_0^p}{dp}\right)\right)\\
\geqslant \;& 1-\mathcal{N}\left(\frac{\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)+\left(\frac{p}{X_0^p}\right)^{\frac{2\mu}{\sigma^2}}\mathcal{N}\left(\frac{-\ln\left(\frac{p}{X_0^p}\right)-\mu T}{\sigma\sqrt{T}}\right)\left(1+\frac{2\mu}{\sigma^2}-\frac{2\mu}{\sigma^2}\frac{p}{X_0^p}\frac{dX_0^p}{dp}\right)\\
\geqslant \;& 0.
\end{aligned}$$

With those calculations we finish this chapter.

7 Summary and outlook

Here we constructed a mathematical framework for the limit order book. That setup is general enough to allow one to include existing empirical observations. Based on the model, we looked at a variety of applications where we worked mainly in a small-investor setup.

Many interesting research question can be identified. A first task is to extend the existing model such that even more practical questions can be answered. Here one can think of topics such as utility maximization or optimal strategies for limit order execution. A second approach would be to go one step back to an agent-based market microstructure model. We will give a short idea of that approach in the next section. Based on different kinds of agents one would want to find a general equilibrium model which leads to a description of the order book.

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Chapter III

Market makers, insiders and limit order traders in a market microstructure model

"Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?"
- Paul R. Halmos

1 Introduction

We consider here a multiperiod market microstructure model. It is an extension of the excellent one-period model by Bondarenko and Sung [BJ03]. Our model has four different types of agents: limit order traders, noise traders, one insider and one market maker where the market maker can choose how much liquidity he provides to the market. The first main feature is that different agents possess different information, in particular, the influence of price and quantities is considered. The trading behaviour of market makers and informed traders is of utmost importance in financial markets and therefore, it is of interest in market microstructure. The market maker usually is the one who provides liquidity to the market. However, many real markets work as hybrid markets, where liquidity is supplied both by the market maker and limit order traders. A rule on the NYSE and Amex states that the specialists have to consider the highest bid and the lowest ask prices in the limit order book, when they determine the price. This has also been a topic of empirical research. In this research, see e.g. Madhavan and Sofianos (1998) [MS98], it has been reported, that liquidity provision significantly depends on the stock which is traded. Some of the stocks basically trade in quote-driven markets, where most of the liquidity is provided by the specialist, where for other stocks, it might be just the other way around. Another study by Harris and Hasbrouck (1996) [HH96b] shows that more than half of the orders submitted through SuperDOT are limit orders. Therefore, one might want to include both limit order traders and specialists in a microstructure model. Models which consider only one of both ways to provide liquidity to the market cannot model the interaction between those two types. One of the most obvious result in our paper in this direction is of course, that we do have an equilibrium under certain conditions once the order book depth is not zero. The interested reader can compare this to the case of strategic market makers when there are no limit order traders present. Results in this area have usually shown the existence of an equilibrium, however there was the additional assumption that there are more than two market makers. With a nontrivial order book, the limit order traders naturally compete with the market maker for liquidity. This has the nice effect that there can be an equilibrium even if there is only one market maker. Of course, this shows up in reality, since there are several exchanges which are organized in a way that there is a monopolist market maker for certain stocks. This specialist can choose whether and how much to use from his own inventory, since the limit order traders come up for the remaining liquidity which needs to be provided.

Furthermore, the insider's trading strategies as well as the specialist's pricing rule are derived endogenously. This extension allows us to thoroughly investigate the influence of time-varying information and the learning behaviour of different agents. All agents are risk-neutral and both the market maker and the informed trader try to maximize their expected profits, whereas the limit order trader's strategy satisfies a no expected profit condition. This allows us to explicitly model the information advantage of the market maker.

Our model is in the spirit of Kyle (1985) [Kyl85] and Bondarenko and Sung (2003) [BJ03]. Bondarenko and Sung (2003) [BJ03] work with a one-period model which we will extend to a multiperiod model. Other papers in this area are Back (1992) [Bac92], Bhattacharya and Spiegel (1991) [BS91], Chung and Charoenwong (1998) [CC98], Dutta and Madhavan (1997) [DM97], Bondarenko (2001) [Bon01], Chakravarty and Holden (1995) [CH95], Chung, Van Ness and Van Ness (1999) [CVNVN99], Dennert (1993) [Den93] and Glosten (1989) [Glo89]. In particular we mention here the model by Bondarenko (2001) [Bon01] . He also works with a multiperiod model, however, he does not include limit order traders and his setup is different: There, the market makers post price schedules in the beginning, before observing the order flow. Therefore he cannot investigate one of our main goals, namely the effects of being able to observe the order flow before posting a price. Furthermore he concludes that there is no equilibrium, when there are less than three market makers, whereas we will show that there exists a (unique) equilibrium even if there is only one market maker. Our general setup also allows us to obtain Kyle's model [Kyl85] in a very straightforward and easy way.

There has been extensive research on trading behaviour of market makers which are perfectly competitive. This leads to a zero-profit condition which greatly simplifies the analysis of those models. However, this contradicts empirical literature quite often, see e.g. Christie and Schultz (1994) [CS94], Christie et al. (1994) [CHS94] or Hasbrouck and Sofianos (1993) [HS93] and Sofianos (1995) [Sof95]. Those findings are supported by both static and dynamic research which investigate the effects of strategic market makers. However, most of those models have been of static nature, and, to the best of our knowledge, there has not been any investigation as far as a dynamic model is concerned, which included both a nontrivial order book and a strategic market maker.

We consider a $N-$period model. In every period the noise traders and the insider submit their orders to the market maker. The market maker then chooses the price after having observed the limit order book and the combined order flow from the insider and the noise traders, such that the market is cleared. A precise definition and conditions on the distribution of the order book depth which are necessary and sufficient for a market equilibrium to exist, will be given later. After the last trading round, the true value of the asset will become known and everyone realizes his gains or losses on the positions which he still has.

We note here the particular role which information plays in this model. The insider knows the true value of the asset, whereas the market maker observes the order book depth and the combined order flow from the insider and the noise traders. This shows that we deal with two kinds of information: One is the information about the asset price, the other is the information on the order flow. It is the interplay between these two aspects which we will investigate and show that both the insider and the market maker will gain from their information whereas the noise traders will lose money in this model. We will show that there exists a market equilibrium and investigate the mutual interdependence of the various parameters. We will also give conditions under which the market equilibrium is unique. Furthermore, we can easily replicate the particular models of Bondarenko and Sung [BJ03] and Kyle [Kyl85] as well as certain well-known results such as the non-existence of an equilibrium if there is no order book and only one competitive market maker, see e.g. Dennert [Den93] or [Bon01]. We consider this unifying framework as one of the main advances of our model.

The remainder of this paper is organized as follows. In the next section we describe the basic model. In Section 3 we give the main theorems. The following section is supposed to describe the four market participants in more detail. Section 4 deals with a predictable order book depth. Next we compare our setup to the case of a competitive market maker, arising from a Bertrand auction. In Section 6 we look at the case of no order book and the one-period case. In the appendix we shall give proofs of the more complicated theorems.

2 The basic model

Throughout the paper, we shall fix a probability space $(\Omega, \mathcal{F}, P)$. In short, the auction mechanism proceeds as follows: At the beginning of every time period, the insider and the noise traders choose their order quantity. Then the market maker observes the combined order flow from the noise traders and the insider as well as the order book. Based on this information he sets the market-clearing price. The setup is as follows: The distribution of the asset value ν at time T is a normal distribution with mean ν_0 and variance σ_0^2. This distribution is known to all participants in the financial market. We consider a $N-$period model from time 0 to T, with the shares being traded at times $0 < t_1 < \cdots < t_N < T$. The length of the n-th period is denoted by Δt_n. The market maker sets the price p_n at time Δt_n.

We model the information through $\sigma-$algebras. Information which is known to the limit order traders just before time t_n is described by the $\sigma-$algebra $\mathcal{F}_{n-1}$ generated by the prices $p_1, \ldots, p_{n-1}$, i.e.

$$\mathcal{F}_{n-1} = \sigma\left(p_1, \ldots, p_{n-1}\right).$$

The limit order traders base their decision at time t_n on the information $\mathcal{F}_{n-1}$.

The order quantities are denoted as follows: The number of shares ordered by the insider are x_n, the noise traders submit z_n, the market maker y_n and the limit order traders $\alpha_n\left(\nu_{n-1}^L - p_n\right)$, where ν_{n-1}^L denotes the limit order traders' expected price of the true asset based on the information they have at time t_n, and α_n is the order book depth. The distribution of the order book depth α_n is known to everyone, however only the market makers observes the exact realization of α_n. We assume that $P\left(\alpha_n > 0\right) = 1$. Of course, this setup implies that the order book has a uniform depth of α_n and limit orders are placed for all prices. The assumption of a linear limit order supply schedule has been used in a number of important market microstructure models see e.g. Kyle (1989) [Kyl89] and Black (1995) [Bla95] and Glosten (1994) [Glo89]. Clearly, it is natural to assume that the limit order traders base their decision on their estimate of the true asset price ν^L_{n-1} as compared to the price p_{n-1}.

Just before the new prices p_n are published, the insider knows all previous prices, his own order flow and the true value of the asset ν, which we describe through the $\sigma-$algebra $\mathcal{I}_{n-1}$ given by

$$\mathcal{I}_{n-1} = \sigma\left(p_1, \ldots, p_{n-1}, x_1, \ldots, x_{n-1}, x_n, \nu\right).$$

The insider bases his decision on the information $\mathcal{I}_{n-1}$.

After the market maker receives the combined order flow from the insider and the noise trader (He can only observe the combined order flow.), he knows all previous prices, the combined order flow from the insider and the noise traders and his own order flow, which leads to the $\sigma-$algebra

$$\mathcal{M}_n = \sigma\left(\alpha_1, \ldots, \alpha_n, p_1, \ldots, p_{n-1}, p_n, x_1 + z_1, \ldots, x_n + z_n, y_1, \ldots, y_n\right).$$

The market maker bases his decision on the information $\mathcal{M}_n$.

The market participants update their information on the true value of the asset in each trading round. Each of the market participants has a different information set. Therefore we have to define the following updated price of the asset at time t_n as follows: The price as seen by the market maker:

$$\nu_n = E\left(\nu \left| \mathcal{M}_n\right.\right).$$

The price as seen by the limit order traders:

$$\nu^L_n = E\left(\nu \left| \mathcal{F}_n\right.\right).$$

Of course, the insider knows the true value:

$$\nu = E\left(\nu \left| \mathcal{I}_n\right.\right).$$

Furthermore we define the updated variance of the true value ν as seen by the market makers at time t_n as

$$\sigma^2_n = Var\left(\nu \left| \mathcal{M}_n\right.\right).$$

The market maker has to ensure that the market clearing condition is satisfied:

$$x_n + y_n + z_n + \alpha_n\left(\nu^L_{n-1} - p_n\right) = 0 \qquad n = 1, 2, \ldots, N. \tag{2.1}$$

We denote the profits of the insider in trading rounds n through N as π^I_n and the profits of the market maker as π^M_n :

$$\begin{aligned}
\pi^I_n &= \sum_{k=n}^{N}\left(\nu - p_k\right) x_k, \\
\pi^M_n &= \sum_{k=n}^{N}\left(\nu - p_k\right) y_k = \sum_{k=n}^{N}\left(p_k - \nu\right)\left(x_k + z_k + \alpha\left(\nu^L_{k-1} - p_n\right)\right).
\end{aligned}$$

We assume that z_n is normally distributed with mean zero and instantaneous variance $\sigma^2_z \Delta t_n$, independent of ν. The distribution of ν is public knowledge (we need this e.g. later for modelling the limit order traders decision on α_n. This assumption allows us to calculate $E\left(\nu \left| \mathcal{M}_n\right.\right)$). We define trading strategies as $N-$tuples:

Definition 2.1 *A trading strategy for the insider is defined as a stochastic process* $X_N = (x_1, \ldots, x_N)$. *A trading strategy for the market maker is defined as a stochastic process* $Y_N = (y_1, \ldots, y_N)$. *Each trader is represented by his trading strategy.*

The vector of order book depths is defined as $\alpha = (\alpha_1, \ldots, \alpha_N)$.

Definition 2.2 (Market Equilibrium) *A security market equilibrium is a triplet* (X_N, Y_N, α) *such that the following conditions are satisfied:*

1. *The market maker maximizes his expected profits:*

$$(y_{n+1}, \ldots, y_N) \in \arg\max E\left(\pi^M_{n+1} \left| \mathcal{M}_{n+1}\right.\right), \qquad n = 0 \ldots N-1.$$

2. *The informed trader maximizes his expected profits:*

$$(x_{n+1}, \ldots, x_N) \in \arg\max E\left(\pi^I_{n+1} \left| \mathcal{I}_n\right.\right), \qquad n = 0 \ldots N-1.$$

3. *The order book depth* α_n *satisfies a no-expected profit condition:*

$$E\left((\nu - p_n)\, \alpha_n \left(\nu^L_{n-1} - p_n\right) \left| \mathcal{F}_{n-1}\right.\right) = 0, \qquad n = 1 \ldots N.$$

4. *The market clearing condition is satisfied:*

$$x_n + y_n + z_n + \alpha_n \left(\nu^L_{n-1} - p_n\right) = 0 \qquad n = 1, \ldots, N.$$

In order to solve for an equilibrium, we have to make several assumptions: First, as customary in the literature on market microstructure, we assume that the quantity x_n traded by the insider at time t_n is linear in the difference between the actual asset price ν and the expected value of p_n, given the information the insider has at time t_n, when he makes his decision. We also assume that the price chosen by the market maker is linear in the total quantity submitted by the insider and the noise trader.

Definition 2.3 (Linear market equilibrium) *A linear market equilibrium is defined as a market equilibrium where* x_n *and* p_n *satisfy in addition*

$$x_n = \beta_n \left(\nu - E\left(p_n \left| \mathcal{I}_{n-1}\right.\right)\right), \Delta t_n \qquad (2.2)$$

$$p_n = p_{n-1} + \lambda_n \left(x_n + z_n\right), \qquad (2.3)$$

where β_n *is* $\mathcal{F}_{n-1}$*−measurable with* $P\left(\beta_n \geqslant 0\right) = 1$ *and* λ_n *is a function of* α_n.

Solutions to the informed trader's and the market maker's optimization problem are called optimal strategies for the informed trader and the market maker respectively.

3 General calculations and main results

The market maker has an informational advantage as compared to the limit order traders, since he observes the combined order flow from the noise traders and the insider. For a model where the market can observe the noise trader order flow separately, see Rochet and Vila (1994) [RV94]. However the market maker also has an informational disadvantage as compared to the insider, since he does not know the true value of the asset. It is this interplay between those two aspects which is one of our main points of investigation. We shall first state our main theorem on conditions for the existence of an equilibrium.

Theorem 3.1 *Necessary conditions for the existence of an equilibrium in this financial market can be described as follows* ($n = 1, \dots, N$)*:*

1. *The insider's strategy at time* t_n *is*
$$x_n = \beta_n \Delta t_n \left(\nu - E\left(p_n \left| \mathcal{I}_{n-1}\right.\right)\right).$$

2. *The market maker sets the price* p_n *at time* t_n *as*
$$p_n = p_{n-1} + \lambda_n \left(\alpha_n\right)\left(x_n + z_n\right).$$

3. *The depth of the market is distributed such that the conditional expected profit of the limit order traders satisfies a zero profit condition*
$$E\left(\left(\nu - p_n\right)\alpha_n\left(\nu_{n-1}^L - p_n\right)\left|\mathcal{F}_{n-1}\right.\right) = 0.$$

4. *The market is cleared, i.e.*
$$0 = x_n + y_n + z_n + \alpha_n\left(\nu_{n-1}^L - p_n\right).$$

The maximal conditional expected profit of the insider is given by

$$\max_{(x_{n+1},\dots,x_N)} E\left(\pi_{n+1}^I \left|\mathcal{I}_n\right.\right) = a_n\left(\nu - p_n\right)^2 + b_n.$$

whereas the maximal conditional expected profit of the market maker is given by

$$\max_{(\lambda_{n+1},\dots,\lambda_N)} E\left(\pi_{n+1}^M \left|\mathcal{M}_{n+1}\right.\right) = c_{n+1}\left(x_{n+1} + z_{n+1}\right)^2 + d_{n+1}\left(x_{n+1} + z_{n+1}\right) + e_{n+1}.$$

The coefficients $a_n, b_n, c_n, d_n, e_n, \beta_n$ are assumed to be $\mathcal{F}_{n-1}$−measurable. Together with α_n and λ_n they satisfy the following recursive system of equations for $n = 1, \dots, N$:

$$a_{n-1} = a_n + \gamma_n\left(1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right) - \gamma_n^2\left(E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) - a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)\right),$$

$$b_{n-1} = a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)\sigma_z^2 \Delta t_n + b_n$$

with the boundary conditions

$$a_N = b_N = 0$$

and

$$\gamma_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n | \mathcal{I}_{n-1})}.$$

The parameter β_n chosen by the informed trader satisfies

$$\frac{2\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n | \mathcal{I}_{n-1})} \left(E(\lambda_n | \mathcal{I}_{n-1}) - a_n E\left(\lambda_n^2 | \mathcal{I}_{n-1}\right)\right) = 1 - 2a_n E(\lambda_n | \mathcal{I}_{n-1})$$

with the second-order conditions given by

$$E(\lambda_n | \mathcal{I}_{n-1}) > a_n E\left(\lambda_n^2 | \mathcal{I}_{n-1}\right) \tag{3.1}$$

The market maker chooses the optimal λ_n given by

$$\begin{aligned} & 2\lambda_n(\alpha_n)(x_n + z_n)\left(\alpha_n - 2c_{n+1}\gamma_{n+1}^2\right) \\ = \; & x_n + z_n + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_n\right) + 2c_{n+1}\gamma_{n+1}^2(p_{n-1} - \nu_n) - d_{n+1}\gamma_{n+1} \end{aligned}$$

where

$$\nu_n = \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} (x_n + z_n - \gamma_n(\nu_{n-1} - p_{n-1}))$$

and

$$\sigma_n^2 = Var(\nu | \mathcal{M}_n) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}.$$

The second-order conditions read as

$$c_{n+1}\gamma_{n+1}^2 - \alpha_n < 0$$

and

$$\alpha_n > 0.$$

The distribution of α_n and λ_n satisfies

$$\begin{aligned} 0 = \; & E(\alpha_n | \mathcal{F}_{n-1})\left(\nu_{n-1}^L - p_{n-1}\right)^2 \\ & + E(\alpha_n \lambda_n(\alpha_n) | \mathcal{F}_{n-1})\gamma_n\left(-E\left(\nu^2 | \mathcal{F}_{n-1}\right) - \left(\nu_{n-1}^L\right)^2 + 4\nu_{n-1}^L p_{n-1} - 2p_{n-1}^2\right) \\ & + E\left(\alpha_n \lambda_n^2(\alpha_n) | \mathcal{F}_{n-1}\right)\left(\gamma_n^2 E\left(\nu^2 | \mathcal{F}_{n-1}\right) + \gamma_n^2 p_{n-1}^2 - 2\gamma_n^2 \nu_{n-1}^L p_{n-1} + \sigma_z^2 \Delta t_n\right). \end{aligned} \tag{3.2}$$

The exact recursive forms of c_n, d_n and e_n are given in the appendix. They satisfy the boundary condition

$$c_{N+1} = d_{N+1} = e_{N+1} = 0$$

Our second theorem concerns the solution of the informed trader's maximization problem.

Theorem 3.2 *Assume that* $E(\lambda_n | \mathcal{I}_{n-1}) \neq a_n E\left(\lambda_n^2 | \mathcal{I}_{n-1}\right)$ *for all* $n = 1, \ldots N$. *Then the informed trader has a unique optimal strategy, whereas the informed trader has several optimal strategies in periods where* $E(\lambda_n | \mathcal{I}_{n-1}) = a_n E\left(\lambda_n^2 | \mathcal{I}_{n-1}\right)$.

Next, we look at the market maker's optimization problem.

Theorem 3.3 *Assume that* $\alpha_n - c_{n+1}\gamma_{n+1}^2 \neq 0$ *for all* $n = 1, \ldots N$. *Then the market maker has a unique optimal strategy, whereas he has several optimal strategies in periods where* $\alpha_n - c_{n+1}\gamma_{n+1}^2 = 0$.

Uniqueness of the equilibrium is provided in the next theorem.

Theorem 3.4 *The market equilibrium is unique if for all* $n = 1, \ldots N$, $\alpha_n - c_{n+1}\gamma_{n+1}^2 \neq 0$, $E(\lambda_n |\mathcal{I}_{n-1}) \neq a_n E(\lambda_n^2 |\mathcal{I}_{n-1})$ *and if the distribution of* α_n *is uniquely characterized by Conditions* (3.1) *and* (3.2).

Now we look at the different market participants separately and we start with a look at the informed trader.

3.1 The informed trader

Assume that β_n is $\mathcal{I}_{n-1}$ measurable and recall that ν is $\mathcal{I}_0-$measurable. We start with writing the quantity x_n as a linear function of ν and p_{n-1}.

Lemma 3.5 *Assume that* ν *is not* $\mathcal{M}_n-$*measurable for every* n. *The quantity* x_n *ordered by the informed trader is linear in the difference between* ν *and* p_{n-1}. *More precisely,*

$$\begin{aligned} x_n &= \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n |\mathcal{I}_{n-1})} (\nu - p_{n-1}) \\ &= : \gamma_n (\nu - p_{n-1}). \end{aligned} \tag{3.3}$$

Proof. We use equations (2.2) and (2.3).

$$\begin{aligned} x_n &= \beta_n \Delta t_n (\nu - p_{n-1} - x_n E(\lambda_n |\mathcal{I}_{n-1})) \\ &\Longrightarrow x_n (1 + \beta_n \Delta t_n E(\lambda_n |\mathcal{I}_{n-1})) = \beta_n \Delta t_n (\nu - p_{n-1}) \\ &\Longrightarrow x_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n |\mathcal{I}_{n-1})} (\nu - p_{n-1}) \end{aligned}$$

provided

$$1 + \beta_n \Delta t_n E(\lambda_n |\mathcal{I}_{n-1}) \neq 0.$$

The condition

$$1 + \beta_n \Delta t_n E(\lambda_n |\mathcal{I}_{n-1}) = 0$$

implies that $\beta_n \Delta t_n (\nu - p_{n-1}) = 0$. This holds if and only if $\beta_n = 0$ or $\nu = p_{n-1}$. However, if $\beta_n = 0$, then we would have a contradiction. The case $\nu = p_{n-1}$ was excluded by assumption. ■

The informed trader knows the realization of ν and chooses the quantity x_n. To solve his optimization problem, we make the following inductive hypothesis:

$$\max_{(x_{n+1},\ldots,x_N)} E(\pi_{n+1}^I |\mathcal{I}_n) = a_n (\nu - p_n)^2 + b_n$$

for some $\mathcal{I}_{n-1}$ measurable functions a_n and b_n with the boundary conditions

$$a_N = b_N = 0$$

This backward iteration procedure approach already appears in Kyle (1985) [Kyl85].

Remark 3.6 *The equation*

$$\max_{(x_{n+1},\ldots,x_N)} E\left(\pi^I_{n+1}\left|\mathcal{I}_n\right.\right) = a_n\left(\nu - p_n\right)^2 + b_n$$

leads to our first interpretation. If the market maker sets a price p_n which is far away from the true asset value ν, the conditional expected profit of the insider is larger than if p_n were close to ν. Of course, this shows how the lack of information on the side of the market maker leads to additional profits on the side of the insider. (Note that we will show later that indeed, $a_n \geqslant 0$ and $b_n \geqslant 0$.)

The informed trader's optimization problem: Choose x_n such as to maximize the expected profit based on the information just before time t_n :

$$\max_{x_n} E\left(\left(\nu - p_n\right)x_n + \pi^I_{n+1}\left|\mathcal{I}_{n-1}\right.\right).$$

Our main result reads as follows.

Theorem 3.7 *Assume that $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) \neq a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$ for all n. Then necessary and sufficient conditions for an equilibrium are*

$$\frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right)} = \frac{1 - 2a_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right)}{2E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) - 2a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)}$$

and

$$E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) > a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$$

Proof. The calculations are shown in the appendix. ■

In the next theorem we look at the case where $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$ for some n.

Theorem 3.8 *Assume that $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$ for some n. Then $a_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = \frac{1}{2}$ is a necessary condition for having an equilibrium. However in this case the strategy of the insider is not unique anymore. If we have $a_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) \neq \frac{1}{2}$ then there is no equilibrium.*

Proof. Again, we refer to the appendix for the details of the proof. ■

Remark 3.9 *This in contrast to the results in Kyle* 1985 *[Kyl85] where we always have a unique equilibrium. If however, as in Kyle* 1985 *[Kyl85], λ_n is constant for all n, we get a unique strategy for the insider, which can be seen from the following lemma.*

Lemma 3.10 *Assume that $\lambda_n \in \mathcal{I}_{n-1}$ for $n = 1, \ldots, N$. The condition $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$ implies that $a_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = \frac{1}{2}$ cannot be true. Therefore we get a unique strategy for the insider. In particular, $\lambda_n \in \mathcal{I}_{n-1}$ if λ_n is constant as in Kyle [Kyl85].*

Proof. Assume that $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$ for some n and therefore $\lambda_n = a_n\lambda_n^2$. Together with $a_n\lambda_n = \frac{1}{2}$ this implies $a_n\lambda_n = 1$ which leads to a contradiction. ■

We now look at the coefficients a_n and b_n.

Lemma 3.11 *Let $n \in \mathbb{N}$ and assume that $E(\lambda_n | \mathcal{I}_{n-1}) \neq a_n E(\lambda_n^2 | \mathcal{I}_{n-1})$. Then the coefficients a_n and b_n are given by the following recursive formula*

$$b_{n-1} = a_n E\left(\lambda_n^2 | \mathcal{I}_{n-1}\right) \sigma_z^2 \Delta t_n + b_n$$

and

$$a_{n-1} = \gamma_n - \gamma_n^2 E(\lambda_n | \mathcal{I}_{n-1}) + a_n + a_n \gamma_n^2 E\left(\lambda_n^2 | \mathcal{I}_{n-1}\right) - 2a_n \gamma_n E(\lambda_n | \mathcal{I}_{n-1})$$

with the boundary conditions $a_N = b_N = 0$. If $E(\lambda_n | \mathcal{I}_{n-1}) = a_n E\left(\lambda_n^2 | \mathcal{I}_{n-1}\right)$ for some $n \in \mathbb{N}$, then necessarily for an equilibrium $a_n E(\lambda_n | \mathcal{I}_{n-1}) = \frac{1}{2}$ and the coefficients a_n and b_n are given by

$$a_{n-1} = a_n$$

and

$$b_{n-1} = E(\lambda_n | \mathcal{I}_{n-1}) \sigma_z^2 \Delta t_n + b_n.$$

Proof. The details of the proof are shown in the appendix. ■

3.2 The market maker

Recall that α_n is $\mathcal{M}_n-$measurable. Assume that λ_n is independent of ν and x_{n-1}. We also assume that ν is independent of α_n.

The following lemma will be used in the sequel.

Lemma 3.12 *We have $x_n + z_n \neq 0$ $P-a.s.$*

Proof. Assume that $P(x_n + z_n = 0) > 0$. Then on this set

$$\beta_n \Delta t_n (\nu - E(p_n | \mathcal{I}_{n-1})) = -z_n.$$

However by the independence of the distributions of ν and z_n, this cannot hold true with positive probability. ■

In the next subsection we want to calculate the conditional expected value of the asset based on the market maker's information. He does not have as much information on ν as the informed trader, however the market maker observes the combined order flow from the liquidity traders and the informed trader. Therefore he does have at least partial information on ν. To make this statement more precise is the purpose of the next subsection.

3.2.1 The conditional stock price value

As a preliminary step, we calculate $E(\nu | \mathcal{M}_n)$

Lemma 3.13 *The expected value of the asset conditional on the market maker's information is given by*

$$\nu_n = E(\nu | \mathcal{M}_n) = \nu_{n-1} + \gamma_n \frac{\sigma_n^2}{\sigma_z^2 \Delta t_{n-1}} (x_n + z_n) - \frac{\sigma_n^2}{\sigma_z^2 \Delta t_{n-1}} \gamma_n^2 (\nu_{n-1} - p_{n-1}).$$

The conditional variance reads as

$$\sigma_n^2 = Var(\nu | \mathcal{M}_n) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}.$$

Proof. Again, the proof is in the appendix. ■

Based on those formulas, we are ready to solve the maximization problem of the marker maker.

3.2.2 Maximization problem of the market maker

As before we proceed by backward induction. We make the following inductive hypotheses:

$$\max_{(\lambda_{n+1},\ldots,\lambda_N)} E\left(\pi_{n+1}^M \left|\mathcal{M}_{n+1}\right.\right) = c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2 + d_{n+1}\left(x_{n+1}+z_{n+1}\right) + e_{n+1}$$

with the boundary conditions

$$c_{N+1} = d_{N+1} = e_{N+1} = 0$$

and the assumption that c_n, d_n and e_n are $\mathcal{M}_{n-1}$−measurable for all n. The market maker maximizes

$$\begin{aligned} & E\left(\left(p_n - \nu\right)\left(x_n + z_n + \alpha_n\left(\nu_{n-1}^L - p_n\right)\right)\left|\mathcal{M}_n\right.\right) \\ & + E\left(c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2 + d_{n+1}\left(x_{n+1}+z_{n+1}\right) + e_{n+1}\left|\mathcal{M}_n\right.\right) \end{aligned} \tag{3.4}$$

The results are summarized in the following two theorems.

Theorem 3.14 *A solution to the market maker's maximization problem exists and the maximizing λ_n is given by*

$$\begin{aligned} & \lambda_n\left(\alpha_n\right)\left(2\left(x_n+z_n\right)\alpha_n - 2c_{n+1}\gamma_{n+1}^2\left(x_n+z_n\right)\right) \\ = \; & x_n + z_n + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_n\right) + 2c_{n+1}\gamma_{n+1}^2 p_{n-1} - d_{n+1}\gamma_{n+1} - 2c_{n+1}\gamma_{n+1}^2\nu_n. \end{aligned}$$

Uniqueness of the solution is given by the following theorem.

Theorem 3.15 *Assume that $\alpha_n - c_{n+1}\gamma_{n+1}^2 \neq 0$. Then the solution to the market maker's maximization problem is unique and the maximizing λ_n is given by*

$$\lambda_n\left(\alpha_n\right) = \frac{x_n + z_n + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_n\right) + 2c_{n+1}\gamma_{n+1}^2 p_{n-1} - d_{n+1}\gamma_{n+1} - 2c_{n+1}\gamma_{n+1}^2\nu_n}{2\left(x_n+z_n\right)\alpha_n - 2c_{n+1}\gamma_{n+1}^2\left(x_n+z_n\right)}.$$

The second-order conditions are

$$c_{n+1}\gamma_{n+1}^2 < \alpha_n$$

and

$$\alpha_n > 0.$$

Proof. As usual, we refer to the appendix for the proof. ■

We also give a condition, when there is no equilibrium.

Lemma 3.16 *Assume that $\alpha_n - c_{n+1}\gamma_{n+1}^2 = 0$ for some n. If in addition,*

$$\left(x_n+z_n\right)\left(x_n + z_n + \alpha_n\nu_{n-1}^L - \alpha_n\nu_n - d_{n+1}\gamma_{n+1}\right) \neq 0$$

then there is no equilibrium. If

$$\left(x_n+z_n\right)\left(x_n + z_n + \alpha_n\nu_{n-1}^L - \alpha_n\nu_n - d_{n+1}\gamma_{n+1}\right) = 0$$

there will be multiple optimal strategies for the market maker.

In the appendix we provide precise recursive formulas for the coefficients c_n, d_n and e_n.

3.3 Noise traders

They submit orders z_n where z_n is normally distributed with expectation 0 and instantaneous variance σ_z^2. Their order size z_n is given by

$$z_n = -x_n - y_n - \alpha_n \left(\nu_{n-1}^L - p_n\right).$$

Their profit reads as

$$\sum_{n=1}^{N} z_n \left(\nu - \nu_{n-1}\right) - \sum_{n=1}^{N} z_n \left(p_n - \nu_{n-1}\right).$$

They do participate, no matter how much they lose or at what level the stock price currently is. A reason for this seemingly paradox behaviour could be liquidity reasons for example, see Kyle (1989) [Kyl89] among others. Since there is extensive literature on that topic, we do not elaborate on the behaviour of the noise traders in any more detail.

3.4 Limit order traders

Here we derive a necessary condition on the distributions of α_n and $\lambda_n(\alpha_n)$ such that the expected profit of the limit order traders satisfies a zero profit condition. The value ν_{n-1}^L denotes the opinion of the limit order traders on the true value of the asset ν at time t_n, just when the market maker decides on the price. This means that the limit order traders buy $\alpha_n \left(\nu_{n-1}^L - p_n\right)$ shares at time t_n where ν_{n-1}^L is given by

$$\nu_{n-1}^L = E\left(\nu \left| \mathcal{F}_{n-1}\right.\right)$$

and $\mathcal{F}_{n-1}$ denotes the information of the limit order trader after trading round $n-1$ but before they know about the price p_n at time t_n.

The limit order traders create the limit order book with depth α_n. At time t_n they buy $\alpha_n \left(\nu_{n-1}^L - p_n\right)$ shares if $\nu_{n-1}^L - p_n$ is positive and sell them if this expression is negative. This means the limit order traders observe the market and update their beliefs on the expected value of ν. In order to find a condition on α_n, in other words, to find out whether and to what extent limit order traders are willing to participate, we assume that their demand $\alpha_n \left(\nu_{n-1}^L - p_n\right)$ satisfies a zero expected profit condition: For each $n = 1, \ldots, N$ we assume that

$$E\left(\left(\nu - p_n\right) \alpha_n \left(\nu_{n-1}^L - p_n\right) \left| \mathcal{F}_{n-1}\right.\right) = 0.$$

Economically, this condition could be justified as follows: At every point in time, there is a large and variable number of limit order traders. If there were any profit opportunities unexploited over the next period, new orders would immediately arise which would drive the expected profit to zero. Our result reads as follows.

Theorem 3.17 *Assume that* $E\left(\left(\nu - p_n\right) \alpha_n \left(\nu_{n-1}^L - p_n\right) \left| \mathcal{F}_{n-1}\right.\right) = 0$. *Then* α_n *and* λ_n *have to satisfy*

$$\begin{aligned} & E\left(\alpha_n \left| \mathcal{F}_{n-1}\right.\right) \left(\left(\nu_{n-1}^L\right)^2 - 2p_{n-1}\nu_{n-1}^L + p_{n-1}^2\right) \\ + \ & E\left(\alpha_n \lambda_n\left(\alpha_n\right) \left| \mathcal{F}_{n-1}\right.\right) \gamma_n \left(-E\left(\nu^2 \left| \mathcal{F}_{n-1}\right.\right) - \left(\nu_{n-1}^L\right)^2 + 4\nu_{n-1}^L p_{n-1} - 2p_{n-1}^2\right) \\ + \ & E\left(\alpha_n \lambda_n^2\left(\alpha_n\right) \left| \mathcal{F}_{n-1}\right.\right) \left(\gamma_n^2 E\left(\nu^2 \left| \mathcal{F}_{n-1}\right.\right) + \gamma_n^2 p_{n-1}^2 - 2\gamma_n^2 \nu_{n-1}^L p_{n-1} + \sigma_z^2 \Delta t_n\right) = 0. \end{aligned}$$

Proof. The proof can be found in the appendix. ■

4 Predictable order book depth

Here we consider the particular case where the order book depth α_n is $\mathcal{F}_{n-1}-$measurable. Furthermore we assume that λ_n is a function of α_n which is $\mathcal{F}_{n-1}-$measurable. Our first lemma shows that $E(p_n|\mathcal{I}_{n-1})$ is $\mathcal{M}_n-$measurable.

Lemma 4.1 *Assume that α_n is independent of $\mathcal{I}_{n-1}$ and λ_n is a deterministic function of α_n. Then $E(p_n|\mathcal{I}_{n-1})$ is $\mathcal{M}_n$-measurable.*

Proof.

$$\begin{aligned}
& E(E(p_n|\mathcal{I}_{n-1})|\mathcal{M}_n) = E(E(p_{n-1} + x_n E(\lambda_n|\mathcal{I}_{n-1})|\mathcal{I}_{n-1})|\mathcal{M}_n) \\
= \ & p_{n-1} + E(x_n E(\lambda_n|\mathcal{I}_{n-1})|\mathcal{M}_n) \\
= \ & E(p_{n-1} + x_n E(\lambda_n) + z_n E(\lambda_n)|\mathcal{M}_n) - z_n E(\lambda_n) \\
= \ & E(p_{n-1} + (x_n + z_n) E(\lambda_n)|\mathcal{M}_n) - z_n E(\lambda_n) \\
= \ & E(p_{n-1} + (x_n + z_n) E(\lambda_n)|\mathcal{M}_n) - z_n E(\lambda_n) \\
= \ & p_{n-1} + (x_n + z_n) E(\lambda_n) = E(p_n|\mathcal{I}_{n-1}).
\end{aligned}$$

■

Next we show that the information of the market maker is exactly one period ahead of the information which the limit order traders have.

Lemma 4.2 *Assume that $\alpha_n \in \mathcal{F}_{n-1}$ and λ_n is a deterministic function of α_n. We claim that*

$$\mathcal{M}_n = \mathcal{F}_n. \tag{4.1}$$

In other words, if the order book depth is nonrandom, then the noise and limit order traders will have the same information as the market maker, only one period later.

Proof. Recall that

$$\mathcal{M}_n = \sigma(\alpha_1, \ldots, \alpha_n, p_1, \ldots, p_n, x_1 + z_1, \ldots, x_n + z_n, y_1, \ldots, y_n)$$

and

$$\mathcal{F}_n = \sigma(p_1, \ldots, p_n),$$

$$\begin{aligned}
x_n &= \beta_n \Delta t_n (\nu - E(p_n|\mathcal{I}_{n-1})), \\
p_n &= p_{n-1} + \lambda_n(\alpha_n)(x_n + z_n), \\
0 &= x_n + y_n + z_n + \alpha_n(\nu_{n-1} - p_n).
\end{aligned}$$

We use equations (2.2) and (2.3) as well as the market clearing condition (2.1). We proceed by induction: $\mathcal{M}_1 = \mathcal{F}_1$: It is clear that $\mathcal{F}_1 \subset \mathcal{M}_1$, so we need to show that $\mathcal{M}_1 \subset \mathcal{F}_1$. By assumption, p_1 is $\mathcal{F}_1-$measurable. This, together with the assumption of F_0-measurable α_1 and $\mathcal{F}_1-$measurable p_0, implies that $x_1 + z_1$ is $\mathcal{F}_1-$measurable. Therefore the market clearing condition implies that y_1 is also $\mathcal{F}_1-$measurable. Now assume that

$$\mathcal{M}_{n-1} = \mathcal{F}_{n-1}.$$

To show $\mathcal{M}_n \subset \mathcal{F}_n$ we proceed as follows: By assumption, p_n and p_{n-1} are $\mathcal{F}_n-$measurable. Then $x_n + z_n$ is $\mathcal{F}_n-$measurable and as before this implies that y_n is $\mathcal{F}_n-$measurable. Here we also used that ν_{n-1} is $\mathcal{F}_{n-1}-$measurable. This leads us to conclude that $\mathcal{M}_n \subset \mathcal{F}_n$. ■

After those preliminary remarks we look at the equations we got in earlier sections and consider the particular case where α_n and λ_n are $\mathcal{F}_{n-1}-$measurable. The previous lemma implies that $\nu_n^L = \nu_n$.

Theorem 4.3 *We have the following necessary conditions for an equilibrium:*

1. *The market clearing condition is given by*

$$x_n + y_n + z_n + \alpha_n (\nu_{n-1} - p_n) = 0.$$

2. *The parameters γ_n and β_n are given by*

$$\begin{aligned} \gamma_n &= \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n \lambda_n}, \\ \beta_n \Delta t_n &= \frac{1}{\lambda_n} - 2a_n. \end{aligned}$$

3. *The second-order conditions are*

$$1 > a_n \lambda_n,$$
$$c_{n+1} \gamma_{n+1}^2 < \alpha_n$$

and

$$\alpha_n > 0.$$

4. *The parameters a_n and b_n are recursively given by*

$$b_{n-1} = a_n \lambda_n^2 \sigma_z^2 \Delta t_n + b_n$$

and

$$a_{n-1} = a_n + \frac{1}{2} \gamma_n (1 - 2a_n \lambda_n)$$

with the boundary conditions

$$a_N = b_N = 0.$$

5. *The parameter λ_n is given by*

$$\lambda_n (\alpha_n) = \frac{x_n + z_n + \alpha_n (\nu_{n-1} - 2p_{n-1} + \nu_n) + 2c_{n+1}\gamma_{n+1}^2 (p_{n-1} - \nu_n) - d_{n+1}\gamma_{n+1}}{2(x_n + z_n)\alpha_n - 2c_{n+1}\gamma_{n+1}^2 (x_n + z_n)}.$$

6. *The conditional expected value of ν can recursively be calculated as*

$$\nu_n = \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} (x_n + z_n - \gamma_n (\nu_{n-1} - p_{n-1}))$$

and the conditional variance is

$$\sigma_n^2 = Var(\nu \,|\mathcal{M}_n) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}.$$

7. *The equations for c_n, d_n and e_n are given in the Appendix.*

8. *The condition on λ_n is as follows*

$$\begin{aligned}
&\nu_{n-1}^2 - 2p_{n-1}\nu_{n-1} + p_{n-1}^2 \\
&+ \lambda_n(\alpha_n)\gamma_n\left(-E\left(\nu^2 \,|\mathcal{F}_{n-1}\right) - \nu_{n-1}^2 + 4\nu_{n-1}p_{n-1} - 2p_{n-1}^2\right) \\
&+ \lambda_n^2(\alpha_n)\left(\gamma_n^2 E\left(\nu^2 \,|\mathcal{F}_{n-1}\right) + \gamma_n^2 p_{n-1}^2 - 2\gamma_n^2\nu_{n-1}p_{n-1} + \sigma_z^2\Delta t_n\right) = 0.
\end{aligned}$$

Proof. The only equations which remain to be shown are the equations for a_n and β_n.

$$\begin{aligned}
a_{n-1} &= a_n + \gamma_n(1 - 2a_n\lambda_n) - \gamma_n^2\left(\lambda_n - a_n\lambda_n^2\right) \\
&= a_n + \frac{1 - 2a_n\lambda_n}{2\lambda_n - 2a_n\lambda_n^2}(1 - 2a_n\lambda_n) - \frac{(1 - 2a_n\lambda_n)^2}{4\left(\lambda_n - a_n\lambda_n^2\right)^2}\left(\lambda_n - a_n\lambda_n^2\right) \\
&= a_n + \frac{(1 - 2a_n\lambda_n)^2}{2\left(\lambda_n - a_n\lambda_n^2\right)} - \frac{(1 - 2a_n\lambda_n)^2}{4\left(\lambda_n - a_n\lambda_n^2\right)} = a_n + \frac{(1 - 2a_n\lambda_n)^2}{4\left(\lambda_n - a_n\lambda_n^2\right)} \\
&= a_n + \frac{1}{2}\gamma_n(1 - 2a_n\lambda_n) = a_n(1 - \gamma_n\lambda_n) + \frac{1}{2}\gamma_n.
\end{aligned}$$

For β_n we obtain

$$\begin{aligned}
\frac{\beta_n\Delta t_n}{1 + \beta_n\Delta t_n\lambda_n} &= \frac{1 - 2a_n\lambda_n}{2\lambda_n - 2a_n\lambda_n^2} \\
&\Longleftrightarrow \beta_n\Delta t_n\left(2\lambda_n - 2a_n\lambda_n^2\right) = (1 + \beta_n\Delta t_n\lambda_n)(1 - 2a_n\lambda_n) \\
&\Longleftrightarrow \beta_n\Delta t_n\left(2\lambda_n - 2a_n\lambda_n^2\right) = 1 - 2a_n\lambda_n + \beta_n\Delta t_n\lambda_n - \beta_n\Delta t_n\lambda_n 2a_n\lambda_n \\
&\Longleftrightarrow \beta_n\Delta t_n\lambda_n = 1 - 2a_n\lambda_n \\
&\Longleftrightarrow \beta_n\Delta t_n = \frac{1}{\lambda_n} - 2a_n.
\end{aligned}$$

■

5 Competitive market maker

5.1 Competitive market maker with order book

Here we modify our previous setup slightly. Quite often in the literature on market microstructure we do not have a maximizing market maker, but instead the maximizing behaviour is replaced by the condition

$$E\left(\nu \,|\mathcal{M}_n\right) = p_n$$

which actually is a zero expected profit condition for the market maker, which we shall show later. For a justification of this in a game-theoretic context see e.g. Kyle (1985) [Kyl85]. We call this the competitive case.

Lemma 5.1 *We have that $\nu_n^L = p_n$, $n = 1, \ldots, N$.*

Proof. The proof uses the assumption

$$E(\nu\,|\mathcal{M}_n) = p_n$$

and the fact that $\mathcal{F}_n \subset \mathcal{M}_n$. Then

$$\nu_n^L = E(\nu\,|\mathcal{F}_n) = E((\nu\,|\mathcal{M}_n)\,|\mathcal{F}_n) = E(p_n\,|\mathcal{F}_n) = p_n.$$

■

The next lemma shows that indeed the condition $E(\nu\,|\mathcal{M}_n) = p_n$ is a zero-profit condition for the market maker.

Lemma 5.2 *The market maker has a zero expected conditional profit.*

Proof. Usually, no formal proof for this is given in the literature. However, we show that even the same technique as before can be used. Therefore we make the inductive hypotheses that

$$\max_{(\lambda_{n+1},\ldots,\lambda_N)} E\left(\pi_{n+1}^M\,|\mathcal{M}_{n+1}\right) = e_{n+1}$$

with $e_{n+1} \in \mathcal{M}_{n+1}$ and the boundary condition $e_{N+1} = 0$. From before we get for the conditional expected profit of the market maker:

$$\begin{aligned}
\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M\,|\mathcal{M}_n\right) &= e_{n+1} + (p_{n-1} - \nu_n)\left(x_n + z_n + \alpha_n \nu_{n-1}^L - \alpha_n p_{n-1}\right) \\
&+ \lambda_n(\alpha_n)(x_n + z_n)^2 + \lambda_n(\alpha_n)(x_n + z_n)\,\alpha_n\left(\nu_{n-1}^L - p_{n-1}\right) \\
&- (p_{n-1} - \nu_n)\,\alpha_n\lambda_n(\alpha_n)(x_n + z_n) - \lambda_n^2(\alpha_n)(x_n + z_n)^2\,\alpha_n.
\end{aligned}$$

Setting $\nu_n = p_n$ and $\nu_{n-1}^L = p_{n-1}$ gives

$$\begin{aligned}
&\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M\,|\mathcal{M}_n\right) = e_{n+1} + (p_{n-1} - p_n)(x_n + z_n + \alpha_n p_{n-1} - \alpha_n p_{n-1}) \\
&+ \lambda_n(\alpha_n)(x_n + z_n)^2 - (p_{n-1} - p_n)\,\alpha_n\lambda_n(\alpha_n)(x_n + z_n) - \lambda_n^2(\alpha_n)(x_n + z_n)^2\,\alpha_n \\
=\;& e_{n+1} - \lambda_n(\alpha_n)(x_n + z_n)^2 + \lambda_n(\alpha_n)(x_n + z_n)^2 \\
&+ \alpha_n\lambda_n^2(\alpha_n)(x_n + z_n)^2 - \alpha_n\lambda_n^2(\alpha_n)(x_n + z_n)^2 = e_{n+1}.
\end{aligned}$$

Of course, the boundary condition $e_{N+1} = 0$ now implies that $e_n = 0$ for all $n = 1, \ldots, N+1$.
■

Theorem 5.3 *The equilibrium in the competitive case is given by the following recursive system of equations*

$$\begin{aligned}
x_n &= \beta_n \Delta t_n\,(\nu - E(p_n\,|\mathcal{I}_{n-1})), \\
p_n &= p_{n-1} + \lambda_n(\alpha_n)(x_n + z_n), \\
0 &= (x_n + z_n)(1 - \alpha_n\lambda_n(\alpha_n)) + y_n
\end{aligned}$$

with

$$E(\nu\,|\mathcal{M}_n) = p_n,$$

$$\sigma_n^2 = Var(\nu\,|\mathcal{M}_n) = \frac{\sigma_{n-1}^2\sigma_z^2\Delta t_{n-1}}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}},$$

$$\lambda_n(\alpha_n) = \frac{\gamma_n \sigma_n^2}{\sigma_z^2 \Delta t_{n-1}},$$

$$\gamma_n \left(p_{n-1}^2 - E\left(\nu^2 \left|\mathcal{F}_{n-1}\right.\right)\right) \left(E\left(\alpha_n \lambda_n(\alpha_n) \left|\mathcal{F}_{n-1}\right.\right) + E\left(\alpha_n \lambda_n^2(\alpha_n) \left|\mathcal{F}_{n-1}\right.\right) \gamma_n\right) + E\left(\alpha_n \lambda_n^2(\alpha_n) \left|\mathcal{F}_{n-1}\right.\right) \sigma_z^2 \Delta t_n = 0,$$

$$\beta_n \Delta t_n \left(2E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) - 2a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)\right) = 1 + \beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) \left(1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right)$$

and the second-order conditions

$$E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) > a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)$$

where

$$\gamma_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)},$$

$$b_{n-1} = a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) \sigma_z^2 \Delta t_n + b_n,$$

$$a_{n-1} = a_n + \gamma_n \left(1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right) - \gamma_n^2 \left(E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) - a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)\right)$$

and the boundary conditions

$$a_N = b_N = 0.$$

Proof. We use that

$$p_n = p_{n-1} + \lambda_n(\alpha_n)(x_n + z_n).$$

Before we have shown that

$$\nu_n = \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} (x_n + z_n)$$

and

$$\sigma_n^2 = Var\left(\nu \left|\mathcal{M}_n\right.\right) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}.$$

This immediately implies

$$\begin{aligned} \lambda_n(\alpha_n) &= \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} \\ &= \frac{\gamma_n \sigma_n^2}{\sigma_z^2 \Delta t_{n-1}}. \end{aligned}$$

The market clearing condition now reads as

$$0 = (x_n + z_n)(1 - \alpha_n \lambda_n(\alpha_n)) + y_n.$$

Rewriting the condition on the order book depth coming from the no expected profit condition for the limit order traders yields

$$\gamma_n E\left(\alpha_n \lambda_n(\alpha_n) \left|\mathcal{F}_{n-1}\right.\right) \left(p_{n-1}^2 - E\left(\nu^2 \left|\mathcal{F}_{n-1}\right.\right)\right) - \gamma_n^2 E\left(\alpha_n \lambda_n^2(\alpha_n) \left|\mathcal{F}_{n-1}\right.\right) \left(p_{n-1}^2 - E\left(\nu^2 \left|\mathcal{F}_{n-1}\right.\right)\right) + E\left(\alpha_n \lambda_n^2(\alpha_n) \left|\mathcal{F}_{n-1}\right.\right) \sigma_z^2 \Delta t_n = 0.$$

■

Lemma 5.4 *In the competitive case, the prices p_n follow a martingale with respect to the filtration $(\mathcal{F}_n)$ and with respect to the filtration $(\mathcal{M}_n)$.*

Proof. Clearly, there is no need to prove that p_n follows a martingale with respect to $\mathcal{M}_n$. For the second part we use Lemma 5.1 and look at

$$\begin{aligned} E\left(p_n \left| \mathcal{F}_{n-1}\right.\right) &= E\left(\left(\nu \left| \mathcal{M}_n\right.\right) \left| \mathcal{F}_{n-1}\right.\right) \\ &= E\left(\nu \left| \mathcal{F}_{n-1}\right.\right) = p_{n-1} \end{aligned}$$

■

5.2 Competitive case with no order book

If we look at the competitive case with no order book, i.e. $\alpha_n = 0$ for $n = 1, \ldots, N$, we obtain a particularly nice result. In the sequel we show that this case can easily be treated with our setup. Since there is no order book, we can assume that $\lambda_n \in \mathcal{F}_{n-1}$. This particular case here will immediately lead to the well-known model by Kyle [Kyl85].

Theorem 5.5 *The equilibrium in the competitive case is given by the following recursive system of equations*

$$\begin{aligned} x_n &= \beta_n \Delta t_n \left(\nu - E\left(p_n \left| \mathcal{I}_{n-1}\right.\right)\right), \\ p_n &= p_{n-1} + \lambda_n\left(\alpha_n\right)\left(x_n + z_n\right), \\ 0 &= x_n + z_n + y_n \end{aligned}$$

with

$$E\left(\nu \left| \mathcal{M}_n\right.\right) = p_n,$$

$$\sigma_n^2 = Var\left(\nu \left| \mathcal{M}_n\right.\right) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}},$$

$$\lambda_n = \frac{\gamma_n \sigma_n^2}{\sigma_z^2 \Delta t_{n-1}} = \frac{1}{\beta_n \Delta t_n}$$

and the second-order conditions

$$1 > a_n \lambda_n$$

where

$$\gamma_n = \frac{\beta_n \Delta t_n}{2},$$

$$b_{n-1} = a_n \lambda_n^2 \sigma_z^2 \Delta t_n + b_n,$$

$$a_{n-1} = \frac{1}{4}\left(\frac{1}{\lambda_n} + a_n\right),$$

and the boundary conditions

$$a_N = b_N = 0.$$

Proof. We use that

$$p_n = p_{n-1} + \lambda_n (x_n + z_n).$$

Before we have shown that

$$\nu_n = \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} (x_n + z_n).$$

and

$$\sigma_n^2 = Var(\nu | \mathcal{M}_n) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}.$$

This immediately implies

$$\lambda_n = \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} = \frac{\gamma_n \sigma_n^2}{\sigma_z^2 \Delta t_{n-1}}.$$

The market clearing condition now reads as

$$0 = x_n + z_n + y_n.$$

The coefficient γ_n is given by

$$\gamma_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n \lambda_n} = \frac{\beta_n \Delta t_n}{2}.$$

Furthermore

$$\begin{aligned} \beta_n \Delta t_n \left(2\lambda_n - 2a_n \lambda_n^2\right) &= 1 + \beta_n \Delta t_n \lambda_n (1 - 2a_n \lambda_n) \\ &\iff 2\beta_n \Delta t_n \lambda_n = 1 + \beta_n \Delta t_n \lambda_n \\ &\iff \beta_n \Delta t_n \lambda_n = 1 \\ &\iff \lambda_n = \frac{1}{\beta_n \Delta t_n}. \end{aligned}$$

Finally we get

$$b_{n-1} = a_n \lambda_n^2 \sigma_z^2 \Delta t_n + b_n$$

and

$$\begin{aligned} a_{n-1} &= a_n + \gamma_n (1 - 2a_n \lambda_n) - \gamma_n^2 \left(\lambda_n - a_n \lambda_n^2\right) \\ &= a_n + \frac{\beta_n \Delta t_n}{2} \left(1 - 2a_n \frac{1}{\beta_n \Delta t_n}\right) - \frac{\beta_n^2 \Delta t_n^2}{4} \left(\frac{1}{\beta_n \Delta t_n} - a_n \frac{1}{\beta_n^2 \Delta t_n^2}\right) \\ &= a_n + \frac{\beta_n \Delta t_n}{2} - a_n - \frac{\beta_n \Delta t_n}{4} + \frac{a_n}{4} = \frac{1}{4} (\beta_n \Delta t_n + a_n) = \frac{1}{4} \left(\frac{1}{\lambda_n} + a_n\right). \end{aligned}$$

■

The careful reader observes that these results are consistent with the results in Kyle (1985) [Kyl85].

6 Two particular cases

6.1 No order book

Here we consider the particular case, when there is no order book at some time t_n. Based on our previous results, there should be no equilibrium. As a necessary condition for an equilibrium to exist, we derived that $\alpha_n > 0$. Therefore, no equilibrium can exist with $\alpha_n = 0$. Here we want to show intuitively how the market maker can make infinite profits in this case and how the other market participants behave.

Theorem 6.1 *If $\alpha_n = 0$ for some $n = 1, \ldots, N$, then the market maker can make infinite profits with probability one.*

Proof. Assume that $\alpha_n = 0$ for some $n = 1, \ldots, N$. The market maker observes the total order flow $x_n + z_n$. If $x_n + z_n > 0$, he chooses λ_n as large as possible. His profit is then $-(x_n + z_n)(\nu - p_n)$ which converges to infinity, as λ_n increases to infinity. Of course, given a strategy at time t_n for the market maker, represented by λ_n, the market maker will always deviate by choosing a larger λ_n. Therefore, no equilibrium can exist. Similarly if $x_n + z_n < 0$, no equilibrium exist. The case $x_n + z_n = 0$ has probability zero and will therefore not be considered by us. ∎

Again, this shows that for the existence of our equilibrium it was crucial to have a nontrivial order book. This result is consistent with earlier results in the literature. In a static model, Dennert [Den93], has already shown a nonexistence result. Even the existence of two market makers is not sufficient to guarantee an equilibrium, see e.g. Bondarenko [Bon01].

6.2 The one-period case

If we choose $N = 1$ we are able to obtain several previous results, known in the literature, in an easy and straightforward way. Again our setup is easy and general enough to allow for particular cases in an extremely convenient way. In particular, we obtain the model by Bondarenko and Sung [BJ03]: We first observe that $\nu_0 = p_0 = \nu_0^L$ and note that we omit the subscript 1 whenever convenient.

Theorem 6.2 *There is an equilibrium in this financial market which is given by*

1. *The insider's strategy at time 1 is*

$$x = \beta(\nu - E(p)).$$

2. *The market maker sets the price p as*

$$p = p_0 + \lambda(\alpha)(x + z).$$

3. *The depth of the market is distributed such that the conditional expected profit of the limit order traders satisfies a zero profit condition*

$$E((\nu - p)\alpha(\nu_0 - p)) = 0.$$

4. *The market is cleared, i.e.*

$$0 = x + y + z + \alpha(\nu_0 - p).$$

The maximal conditional expected profit of the insider is given by

$$\max_x E\left(\pi^I | \mathcal{I}\right) = a(\nu - p)^2 + b.$$

whereas the maximal conditional expected profit of the market maker yields

$$\max_\lambda E\left(\pi^M | \mathcal{M}_1\right) = c(x+z)^2 + d(x+z) + e.$$

The coefficients a, b, c, d, e, β are $\mathcal{F}_0-$measurable and together with α, λ satisfy the following recursive system of equations:

$$a = \gamma - \gamma^2 E(\lambda),$$
$$b = 0$$

where

$$\gamma = \frac{\beta}{1 + \beta E(\lambda)}.$$

The parameter β chosen by the informed trader satisfies

$$\frac{2\beta}{1 + \beta E(\lambda)} E(\lambda) = 1$$

with the second-order conditions given by

$$E(\lambda) > 0.$$

The market maker chooses the optimal λ given by

$$2\lambda(\alpha)(x+z)\alpha = x + z + \alpha(\nu_1 - p_0)$$

where

$$\nu_1 = p_0 + \frac{\gamma\sigma_0^2}{\gamma^2\sigma_0^2 + \sigma_z^2}(x+z)$$

and

$$\begin{aligned} \sigma_1^2 &= Var(\nu | \mathcal{M}_1) = \frac{\sigma_0^2\sigma_z^2}{\gamma^2\sigma_0^2 + \sigma_z^2}, \\ E\left(\nu^2\right) &= \sigma_0^2 + p_0^2. \end{aligned}$$

The second-order condition reads as

$$\alpha > 0.$$

The distribution of α and λ satisfies

$$\begin{aligned} & E(\alpha\lambda(\alpha))\gamma\left(-E\left(\nu^2\right) + p_0^2\right) \\ + \; & E\left(\alpha\lambda^2(\alpha)\right)\left(\gamma^2 E\left(\nu^2\right) - \gamma^2 p_0^2 + \sigma_z^2\right) = 0. \end{aligned}$$

We want to show that this model coincides with the model by Bondarenko and Sung [BJ03].

Theorem 6.3 *The model in this subsection leads to the same results as the model by Bondarenko and Sung [BJ03]*

Proof. The proof is straightforward and therefore we omit the calculations. ∎

Furthermore, we also obtain a formula for the expected profit of the market maker, which is not in the paper by Bondarenko and Sung [BJ03] and which is given by

$$\max_{\lambda} E\left(\pi^M \left|\mathcal{M}_1\right.\right) = c\,(x+z)^2$$

where

$$\begin{aligned}
c &= \frac{1}{4\alpha} + \frac{\alpha\left(\frac{\gamma\sigma_0^2}{\gamma^2\sigma_0^2+\sigma_z^2}\right)^2}{4} - \frac{1}{2}\frac{\gamma\sigma_0^2}{\gamma^2\sigma_0^2+\sigma_z^2} \\
&= \frac{1}{4\alpha} + \frac{\alpha\left(\frac{\gamma}{\gamma^2+\rho^2}\right)^2}{4} - \frac{1}{2}\frac{\gamma}{\gamma^2+\rho^2} = \frac{1}{4\alpha} + \frac{\alpha\left(\frac{\gamma}{\gamma^2+\rho^2}\right)^2}{4} - \frac{1}{2}\frac{\gamma}{\gamma^2+\rho^2} \\
&= \frac{1}{4\alpha} + \frac{\alpha}{4\theta^2} - \frac{1}{2\theta} = \frac{1}{4\alpha} + \frac{\alpha}{4\theta^2} - \frac{1}{2\theta} = \frac{\theta^2}{4\alpha\theta^2} + \frac{\alpha^2}{4\alpha\theta^2} - \frac{2\alpha\theta}{4\alpha\theta^2} = \frac{\theta^2+\alpha^2-2\alpha\theta}{4\alpha\theta^2} \\
&= \frac{(\theta-\alpha)^2}{4\alpha\theta^2} \geqslant 0
\end{aligned}$$

Intuitively, the further θ and α are apart, the larger is the expected profit of the market maker.

7 Summary and outlook

In this section we investigated a dynamic market microstructure model with four different kinds of agents. The model was set up in such a way that it generalized existing approaches. Furthermore, some of the standard results and models in market microstructure could now be derived in a very straightforward way.

As extension of this research one can think of many different topics. First of all, one would look to relax some of the assumptions to have a more general model. Furthermore, it would be useful to identify also other kinds of agents and include them in the setup. It would also be worthwhile investigating of how one could give a unifying framework for other models which exist in the market microstructure literature. Here we worked in a discrete time setting. Further research questions arise if we consider a continuous time model. There have been some considerations in the literature, however one could also include limit order traders as well as other agents in a continuous setting.

Appendix

A The informed trader

Before starting with the main calculations, we state and prove a preliminary lemma.

Lemma A.1 *ν is not $\mathcal{M}_n$−measurable for all n.*

Proof. Recall that

$$\mathcal{M}_n = \sigma\left(\alpha_1, \ldots, \alpha_n, p_1, \ldots, p_{n-1}, p_n, x_1 + z_1, \ldots, x_n + z_n, y_1, \ldots, y_n\right)$$

and that z_n is independent of both ν and $\alpha_k, 1 \leqslant k \leqslant n$. From this we immediately see that $\nu \notin \mathcal{M}_n$. ∎

Remark A.2 *Clearly, the previous result is necessary in order for us to be able to set up the model the way we wanted to: The insider should really be the only one who knows the exact realization of ν.*

Here we want to prove Theorems 3.7, 3.8 as well as Lemma 3.11. We start with a preliminary lemma.

Lemma A.3 *From the assumption*

$$\max_{(x_{n+1},\ldots,x_N)} E\left(\pi^I_{n+1} \left|\mathcal{I}_n\right.\right) = a_n\left(\nu - p_n\right)^2 + b_n$$

we get

$$\begin{aligned} & \max_{(x_{n+1},\ldots,x_N)} E\left(\pi^I_{n+1} \left|\mathcal{I}_{n-1}\right.\right) \\ = & \max_{(x_{n+1},\ldots,x_N)} E\left(E\left(\pi^I_{n+1} \left|\mathcal{I}_n\right.\right) \left|\mathcal{I}_{n-1}\right.\right) = \max_{(x_{n+1},\ldots,x_N)} E\left(a_n\left(\nu - p_n\right)^2 + b_n \left|\mathcal{I}_{n-1}\right.\right). \end{aligned}$$

Proof of Theorem 3.8

The informed trader solves

$$\begin{aligned} & \max_{(x_n,\ldots,x_N)} E\left(\left(\nu - p_n\right)x_n + \pi^I_{n+1} \left|\mathcal{I}_{n-1}\right.\right) \\ = & \max_{x_n} E\left(\left(\nu - p_n\right)x_n + a_n\left(\nu - p_n\right)^2 + b_n \left|\mathcal{I}_{n-1}\right.\right). \end{aligned}$$

We shall assume that a_n and b_n are $\mathcal{I}_{n-1}$−measurable. Then

$$\begin{aligned}
& \max_{(x_n,\dots,x_N)} E\left((\nu - p_n)\, x_n + \pi^I_{n+1} \left|\mathcal{I}_{n-1}\right.\right) \\
= \; & \max_{x_n} E\left((\nu - p_{n-1} - \lambda_n (x_n + z_n))\, x_n + a_n (\nu - p_{n-1} - \lambda_n (x_n + z_n))^2 + b_n \left|\mathcal{I}_{n-1}\right.\right) \\
= \; & \max_{x_n} (\nu - p_{n-1} - E(\lambda_n \left|\mathcal{I}_{n-1}\right.) x_n)\, x_n + a_n (\nu - p_{n-1})^2 + a_n x_n^2 E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) \\
+ \; & a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) \sigma_z^2 \Delta t_n - 2 a_n (\nu - p_{n-1})\, x_n E(\lambda_n \left|\mathcal{I}_{n-1}\right.) + b_n.
\end{aligned}$$

Note that we implicitly used that

$$E(z_n \left|\mathcal{I}_{n-1}\right.) = 0$$

and

$$E\left(z_n^2 \left|\mathcal{I}_{n-1}\right.\right) = E\left(z_n^2\right) = \sigma_z^2 \Delta t_n$$

since z_n is independent of $\mathcal{I}_{n-1}$. In a first step we assume that $E(\lambda_n \left|\mathcal{I}_{n-1}\right.) \neq a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)$. Looking at the first order conditions and differentiating the following expression with respect to x_n

$$\begin{aligned}
& (\nu - p_{n-1} - E(\lambda_n \left|\mathcal{I}_{n-1}\right.) x_n)\, x_n + a_n (\nu - p_{n-1})^2 \\
+ \; & a_n x_n^2 E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) + a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) \sigma_z^2 \Delta t_n - 2 a_n (\nu - p_{n-1})\, x_n E(\lambda_n \left|\mathcal{I}_{n-1}\right.) + b_n
\end{aligned}$$

yields

$$\begin{aligned}
0 \quad &= \quad \nu - p_{n-1} - 2E(\lambda_n \left|\mathcal{I}_{n-1}\right.) x_n + 2 a_n x_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) \\
& \quad\; - 2 a_n (\nu - p_{n-1})\, E(\lambda_n \left|\mathcal{I}_{n-1}\right.) \\
& \iff \quad x_n = \frac{-1 + 2 a_n E(\lambda_n \left|\mathcal{I}_{n-1}\right.)}{-2E(\lambda_n \left|\mathcal{I}_{n-1}\right.) + 2 a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)} (v - p_{n-1}).
\end{aligned}$$

Therefore, using Lemma 3.5 we get

$$\frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n \left|\mathcal{I}_{n-1}\right.)} = \frac{-1 + 2 a_n E(\lambda_n \left|\mathcal{I}_{n-1}\right.)}{-2E(\lambda_n \left|\mathcal{I}_{n-1}\right.) + 2 a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)}.$$

The second-order conditions yield

$$\begin{aligned}
-2E(\lambda_n \left|\mathcal{I}_{n-1}\right.) + 2 a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) \quad & < \quad 0 \\
& \iff \quad E(\lambda_n \left|\mathcal{I}_{n-1}\right.) - a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) > 0.
\end{aligned}$$

In the next theorem we look at the case where $E(\lambda_n \left|\mathcal{I}_{n-1}\right.) = a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)$ for some n, i.e. we prove Theorem 3.8.

Proof of Theorem 3.8

Consider some n for which

$$E(\lambda_n \left|\mathcal{I}_{n-1}\right.) = a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right).$$

Then the first order conditions are zero if and only if

$$\begin{aligned}\nu - p_{n-1} - 2a_n (\nu - p_{n-1}) E(\lambda_n | \mathcal{I}_{n-1}) &= 0 \\ \Longleftrightarrow \quad & (\nu - p_{n-1})(1 - 2a_n E(\lambda_n | \mathcal{I}_{n-1})) = 0 \\ \Longleftrightarrow \quad & \nu = p_{n-1} \text{ or } a_n E(\lambda_n | \mathcal{I}_{n-1}) = \frac{1}{2}.\end{aligned}$$

The first condition implies $\nu \in \mathcal{M}_{n-1}$. This means that ν is known to the market maker after round $n-1$. We discussed this case before and excluded it. From the second condition

$$a_n E(\lambda_n | \mathcal{I}_{n-1}) = \frac{1}{2}$$

we get with

$$E(\lambda_n | \mathcal{I}_{n-1}) = a_n E(\lambda_n^2 | \mathcal{I}_{n-1})$$

that

$$\frac{1}{2} = a_n^2 E(\lambda_n^2 | \mathcal{I}_{n-1}).$$

Here the first order conditions are always zero. Therefore

$$\begin{aligned}&\max_{x_n} (\nu - p_{n-1} - E(\lambda_n | \mathcal{I}_{n-1}) x_n) x_n + a_n (\nu - p_{n-1})^2 + a_n x_n^2 E(\lambda_n^2 | \mathcal{I}_{n-1}) \\ &+ a_n E(\lambda_n^2 | \mathcal{I}_{n-1}) \sigma_z^2 \Delta t_n - 2a_n (\nu - p_{n-1}) x_n E(\lambda_n | \mathcal{I}_{n-1}) + b_n \\ = \; & a_n (\nu - p_{n-1})^2 + E(\lambda_n | \mathcal{I}_{n-1}) \sigma_z^2 \Delta t_n + b_n.\end{aligned}$$

From this we set

$$a_{n-1} = a_n$$

and

$$b_{n-1} = E(\lambda_n | \mathcal{I}_{n-1}) \sigma_z^2 \Delta t_n + b_n.$$

Now, in period t_n we therefore have several choices of x_n. The conditional expected profit of the insider will be the same, no matter what x_n he chooses. Therefore the strategy of the insider is not unique anymore. We want to add a small remark: The condition $a_n E(\lambda_n | \mathcal{I}_{n-1}) = \frac{1}{2}$ cannot be satisfied in the last trading period since $a_N = 0$.

For the case $E(\lambda_n | \mathcal{I}_{n-1}) \neq a_n E(\lambda_n^2 | \mathcal{I}_{n-1})$ for all $n = 1, \dots, N$ we have a closer look at the maximal expected conditional profit. This allows us to compute the recursive formulas for a_n and b_n.

$$\begin{aligned}&\max_{(x_n,\dots,x_N)} E\left((\nu - p_n) x_n + \pi_{n+1}^I | \mathcal{I}_{n-1}\right) \\ = \; & \max_{x_n} (\nu - p_{n-1} - E(\lambda_n | \mathcal{I}_{n-1}) x_n) x_n + a_n (\nu - p_{n-1})^2 \\ &+ a_n x_n^2 E(\lambda_n^2 | \mathcal{I}_{n-1}) + a_n E(\lambda_n^2 | \mathcal{I}_{n-1}) \sigma_z^2 \Delta t_n - 2a_n (\nu - p_{n-1}) x_n E(\lambda_n | \mathcal{I}_{n-1}) + b_n \\ = \; & (\nu - p_{n-1})^2 \left(\gamma_n - \gamma_n^2 E(\lambda_n | \mathcal{I}_{n-1}) + a_n + a_n \gamma_n^2 E(\lambda_n^2 | \mathcal{I}_{n-1}) - 2a_n \gamma_n E(\lambda_n | \mathcal{I}_{n-1})\right) \\ &+ a_n E(\lambda_n^2 | \mathcal{I}_{n-1}) \sigma_z^2 \Delta t_n + b_n.\end{aligned}$$

Recall that

$$\gamma_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n | \mathcal{I}_{n-1})}.$$

From this we set

$$b_{n-1} = a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) \sigma_z^2 \Delta t_n + b_n$$

and

$$\begin{aligned} a_{n-1} =& \gamma_n - \gamma_n^2 E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) + a_n + a_n \gamma_n^2 E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) - 2 a_n \gamma_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) \\ =& a_n + \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)} \left(1 - 2 a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right) \\ & - \left(\frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)}\right)^2 \left(E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) - a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)\right). \end{aligned}$$

Clearly, this implies that a_n and b_n are indeed $\mathcal{I}_{n-1}$−measurable.

We consider two more cases. First

$$\left(\nu - p_{n-1}\right)\left(1 - 2 a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right) < 0.$$

Then the insider chooses the quantity minus infinity, so there is no equilibrium. Second

$$\left(\nu - p_{n-1}\right)\left(1 - 2 a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right) > 0.$$

Then the insider chooses a quantity plus infinity and he gains an infinite amount of money. Then y_n would be minus infinity and the market maker would make an infinite loss. Of course the market maker can deviate from this strategy by picking a different λ_n. Therefore he will do so and this case cannot occur in equilibrium. Summing up, we see that λ_n will not be chosen such that $E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)$, except if in addition $a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) = \frac{1}{2}$.

B The market maker

Assume that α_n, β_n and λ_n are $\mathcal{M}_n$−measurable. Furthermore assume that λ_n is independent of ν and x_{n-1}. We also assume that ν is independent of α_n. First we start with proving the formulas for ν_n and σ_n^2 as in Theorem 3.13.

Proof of Theorem 3.13

We look at

$$x_n + z_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)} \left(\nu - p_{n-1}\right) + z_n.$$

Assume that x_n is $\mathcal{M}_n$−measurable. Then

$$x_n + z_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)} \left(\nu - p_{n-1}\right) + z_n.$$

We assume that β_n is $\mathcal{M}_{n-1}$−measurable and independent of ν. We start with the conditional expected value of ν.

$$\begin{aligned} E\left(\nu \left|\mathcal{M}_n\right.\right) &= E\left(v \left|\mathcal{M}_{n-1}\right.\right) + \frac{Cov\left(\nu, x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)}{Var\left(x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)} \left(x_n + z_n - E\left(x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)\right) \\ &= \nu_{n-1} + \frac{Cov\left(\nu, x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)}{Var\left(x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)} \left(x_n + z_n - E\left(x_n \left|\mathcal{M}_{n-1}\right.\right)\right). \end{aligned}$$

First we calculate

$$
\begin{aligned}
Cov\left(\nu, x_n + z_n \left|\mathcal{M}_{n-1}\right.\right) &= Cov\left(\nu, \frac{\beta_n \Delta t_n}{1+\beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)}\left(\nu - p_{n-1}\right) + z_n \left|\mathcal{M}_{n-1}\right.\right) \\
&= Cov\left(\nu, \gamma_n\left(\nu - p_{n-1}\right) + z_n \left|\mathcal{M}_{n-1}\right.\right) = Cov\left(\nu, \gamma_n\left(\nu - p_{n-1}\right)\left|\mathcal{M}_{n-1}\right.\right) = \gamma_n \sigma_{n-1}^2
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left(x_n + z_n \left|\mathcal{M}_{n-1}\right.\right) &= Var\left(\frac{\beta_n \Delta t_n}{1+\beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)}\left(\nu - p_{n-1}\right) + z_n \left|\mathcal{M}_{n-1}\right.\right) \\
&= Var\left(\gamma_n\left(\nu - p_{n-1}\right) + z_n \left|\mathcal{M}_{n-1}\right.\right) \\
&= Var\left(\gamma_n\left(\nu - p_{n-1}\right) + z_n \left|\mathcal{M}_{n-1}\right.\right) = \gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}.
\end{aligned}
$$

We assume that $\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1} \neq 0$ (This is clearly satisfied since $\sigma_z^2 > 0$ by assumption.). Using this, we can calculate

$$
\begin{aligned}
E\left(\nu \left|\mathcal{M}_n\right.\right) &= \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}\left(x_n + z_n - \gamma_n\left(\nu_{n-1} - p_{n-1}\right)\right) \\
&= \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}\left(x_n + z_n - \gamma_n\left(\nu_{n-1} - p_{n-1}\right)\right)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\sigma_n^2 &= Var\left(\nu \left|\mathcal{M}_n\right.\right) = Var\left(\nu \left|\mathcal{M}_{n-1}\right.\right) - \frac{\left(Cov\left(\nu, x_n + z_n\right)\left|\mathcal{M}_{n-1}\right.\right)^2}{Var\left(x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)} \\
&= \frac{\gamma_n^2 \sigma_{n-1}^4 + \sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1} - \gamma_n^2 \sigma_{n-1}^4}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}.
\end{aligned}
$$

This leads to

$$
\nu_n = \nu_{n-1} + \gamma_n \frac{\sigma_n^2}{\sigma_z^2 \Delta t_{n-1}}\left(x_n + z_n\right) - \frac{\sigma_n^2}{\sigma_z^2 \Delta t_{n-1}} \gamma_n^2\left(\nu_{n-1} - p_{n-1}\right)
$$

since

$$
\gamma_n \frac{\sigma_n^2}{\sigma_z^2 \Delta t_{n-1}} = \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}.
$$

Next we give proofs for Theorems 3.14 and 3.15.

Proofs of Theorems 3.14 **and** 3.15

We make the following inductive hypotheses:

$$
\max_{\left(\lambda_{n+1}, \ldots, \lambda_N\right)} E\left(\pi_{n+1}^M \left|\mathcal{M}_{n+1}\right.\right) = c_{n+1}\left(x_{n+1} + z_{n+1}\right)^2 + d_{n+1}\left(x_{n+1} + z_{n+1}\right) + e_{n+1}
$$

with the boundary conditions

$$
c_{N+1} = d_{N+1} = e_{N+1} = 0
$$

and the assumption that c_{n+1}, d_{n+1} and e_{n+1} are $\mathcal{M}_n$−measurable. The market maker solves

$$\max_{\lambda_n}\left[E\left(\left(p_n-\nu\right)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-p_n\right)\right)|\mathcal{M}_n\right)\right.$$
$$\left.+E\left(c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2+d_{n+1}\left(x_{n+1}+z_{n+1}\right)+e_{n+1}|\mathcal{M}_n\right)\right].$$

We start with assuming that $\alpha_n - c_{n+1}\gamma_{n+1}^2 \neq 0$ and proceed in two steps: First we calculate

$$\begin{aligned} & E\left(c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2+d_{n+1}\left(x_{n+1}+z_{n+1}\right)+e_{n+1}|\mathcal{M}_n\right) \\ = \; & c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\nu_n^2\right)-2p_nc_{n+1}\gamma_{n+1}^2\nu_n+c_{n+1}\gamma_{n+1}^2p_n^2 \\ & + d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_n+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}. \end{aligned}$$

Plugging in the equation for p_n,

$$p_n = p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)$$

yields

$$\begin{aligned} & E\left(c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2+d_{n+1}\left(x_{n+1}+z_{n+1}\right)+e_{n+1}|\mathcal{M}_n\right) \\ = \; & c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\nu_n^2\right)-2\left(p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)c_{n+1}\gamma_{n+1}^2\nu_n \\ & + c_{n+1}\gamma_{n+1}^2\left(p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)^2+d_{n+1}\gamma_{n+1}\nu_n \qquad \text{(B.1)} \\ & - d_{n+1}\gamma_{n+1}\left(p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}. \end{aligned}$$

Differentiating with respect to λ_n leads to:

$$\begin{aligned} & \frac{\partial}{\partial\lambda_n}E\left(c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2+d_{n+1}\left(x_{n+1}+z_{n+1}\right)+e_{n+1}|\mathcal{M}_n\right) \\ = \; & -2\left(x_n+z_n\right)c_{n+1}\gamma_{n+1}^2\nu_n+2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)\left(x_n+z_n\right) \qquad \text{(B.2)} \\ & - d_{n+1}\gamma_{n+1}\left(x_n+z_n\right). \end{aligned}$$

To continue, we assume that α_{n+1} is independent of $\mathcal{M}_n$. In a second step we calculate

$$\begin{aligned} & E\left(p_{n-1}\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-p_{n-1}-\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)\right)|\mathcal{M}_n\right) \\ & +E\left(\left(\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)-\nu\right)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-p_{n-1}-\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)\right)|\mathcal{M}_n\right) \\ = \; & \left(p_{n-1}-\nu_n\right)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_np_{n-1}\right)+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)^2 \qquad \text{(B.3)} \\ & + \lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\alpha_n\left(\nu_{n-1}^L-p_{n-1}\right)-\left(p_{n-1}-\nu_n\right)\alpha_n\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right) \\ & - \lambda_n^2\left(\alpha_n\right)\left(x_n+z_n\right)^2\alpha_n. \end{aligned}$$

(Clearly, $\nu_{n-1}^L = E\left(\nu|\mathcal{F}_{n-1}\right)$ is $\mathcal{M}_n$−measurable.) Differentiating this last expression with respect to λ_n yields

$$\begin{aligned} & \left(x_n+z_n\right)^2+\left(x_n+z_n\right)\alpha_n\left(\nu_{n-1}^L-p_{n-1}\right) \qquad \text{(B.4)} \\ - \; & \left(p_{n-1}-\nu_n\right)\alpha_n\left(x_n+z_n\right)-2\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)^2\alpha_n. \end{aligned}$$

Since the market maker can always choose to stop trading, we have to ensure that this term has a finite maximum. Therefore we take the second derivative, which yields

$$-2\left(x_n+z_n\right)^2\alpha_n.$$

This has be to negative, which is equivalent to the condition $\alpha_n > 0$ (with probability one). We add the term $(B.3)$ to the expression $(B.1)$ and set the derivative of the sum to zero.

$$\begin{aligned}
&(x_n+z_n)^2+(x_n+z_n)\,\alpha_n\left(\nu^L_{n-1}-p_{n-1}\right)-(p_{n-1}-\nu_n)\,\alpha_n\,(x_n+z_n)\\
&-2\lambda_n(\alpha_n)\,(x_n+z_n)^2\,\alpha_n-2\,(x_n+z_n)\,c_{n+1}\gamma^2_{n+1}\nu_n\\
&+2c_{n+1}\gamma^2_{n+1}\,(p_{n-1}+\lambda_n(\alpha_n)\,(x_n+z_n))\,(x_n+z_n)-d_{n+1}\gamma_{n+1}\,(x_n+z_n)=0.
\end{aligned}$$

Solving this for λ_n yields

$$\lambda_n(\alpha_n)=\frac{x_n+z_n+\alpha_n\left(\nu^L_{n-1}-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma^2_{n+1}\,(p_{n-1}-\nu_n)-d_{n+1}\gamma_{n+1}}{2\,(x_n+z_n)\,\alpha_n-2c_{n+1}\gamma^2_{n+1}\,(x_n+z_n)}. \tag{B.5}$$

Multiplying by x_n+z_n and $(x_n+z_n)^2$ respectively, we obtain

$$\begin{aligned}
&\lambda_n(\alpha_n)\,(x_n+z_n)\\
=\;&\frac{x_n+z_n+\alpha_n\left(\nu^L_{n-1}-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma^2_{n+1}\,(p_{n-1}-\nu_n)-d_{n+1}\gamma_{n+1}}{2\alpha_n-2c_{n+1}\gamma^2_{n+1}}
\end{aligned}$$

and

$$\begin{aligned}
&\lambda^2_n(\alpha_n)\,(x_n+z_n)^2\\
=\;&\frac{\left(x_n+z_n+\alpha_n\left(\nu^L_{n-1}-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma^2_{n+1}\,(p_{n-1}-\nu_n)-d_{n+1}\gamma_{n+1}\right)^2}{\left(2\alpha_n-2c_{n+1}\gamma^2_{n+1}\right)^2}.
\end{aligned}$$

Further simplifying and plugging in $\lambda_n(\alpha_n)$ from Equation $(B.5)$:

$$\begin{aligned}
&\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi^M_n\,|\mathcal{M}_n\right)\\
=\;&\frac{\left(x_n+z_n+\alpha_n\left(\nu^L_{n-1}-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma^2_{n+1}\,(p_{n-1}-\nu_n)-d_{n+1}\gamma_{n+1}\right)^2}{4\alpha_n-4c_{n+1}\gamma^2_{n+1}}\\
&+\;(p_{n-1}-\nu_n)\left(x_n+z_n+\alpha_n\nu^L_{n-1}-\alpha_n p_{n-1}\right)+c_{n+1}\gamma^2_{n+1}\left(\sigma^2_n+\nu^2_n\right)\\
&-\;2p_{n-1}c_{n+1}\gamma^2_{n+1}\nu_n+\;c_{n+1}\gamma^2_{n+1}p^2_{n-1}+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}\\
&+\;c_{n+1}\sigma^2_z\Delta t_{n+1}+e_{n+1}.
\end{aligned}$$

Now plugging in the expression for ν_n,

$$\nu_n=\nu_{n-1}+\frac{\gamma_n\sigma^2_{n-1}}{\gamma^2_n\sigma^2_{n-1}+\sigma^2_z\Delta t_{n-1}}\,(x_n+z_n-\gamma_n\,(\nu_{n-1}-p_{n-1}))$$

and looking at the terms with x_n+z_n, one can see that we can write

$$\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi^M_n\,|\mathcal{M}_n\right)=c_n\,(x_n+z_n)^2+d_n\,(x_n+z_n)+e_n$$

for some $\mathcal{M}_n-$measurable functions c_n, d_n and e_n. From this calculation one also gets that indeed $c_{n+1}, d_{n+1}, e_{n+1}$ are $\mathcal{M}_n-$measurable. Now we look at the second-order conditions. We differentiate the sum of $(B.2)$ and $(B.4)$ again with respect to λ_n :

$$\begin{aligned}
&(x_n+z_n)^2+(x_n+z_n)\,\alpha_n\left(\nu^L_{n-1}-p_{n-1}\right)-(p_{n-1}-\nu_n)\,\alpha_n\,(x_n+z_n)\\
&-2\lambda_n(\alpha_n)\,(x_n+z_n)^2\,\alpha_n-2\,(x_n+z_n)\,c_{n+1}\gamma^2_{n+1}\nu_n\\
&+2c_{n+1}\gamma^2_{n+1}\,(p_{n-1}+\lambda_n(\alpha_n)\,(x_n+z_n))\,(x_n+z_n)-d_{n+1}\gamma_{n+1}\,(x_n+z_n)
\end{aligned}$$

which yields

$$\begin{aligned} -2\left(x_n+z_n\right)^2\alpha_n+2c_{n+1}\gamma_{n+1}^2\left(x_n+z_n\right)^2 &< 0 \\ &\Longleftrightarrow\quad c_{n+1}\gamma_{n+1}^2<\alpha_n. \end{aligned}$$

This finishes the proof of Theorems 3.14 and 3.15.

In the following we want to prove Lemma 3.16.

Proof of Lemma 3.16

We have a look at the maximal expected conditional profit.

$$\begin{aligned} &\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M\left|\mathcal{M}_n\right.\right) \\ = \;& \lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L+\nu_n\right)-2\alpha_n\nu_n-d_{n+1}\gamma_{n+1}\right) \\ &+\;\left(p_{n-1}-\nu_n\right)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+\alpha_n\left(\sigma_n^2+\nu_n^2\right)-2p_{n-1}\alpha_n\nu_n \\ &+\;\alpha_n p_{n-1}^2+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}. \end{aligned}$$

We consider three cases. First

$$\left(x_n+z_n\right)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L+\nu_n\right)-2\alpha_n\nu_n-d_{n+1}\gamma_{n+1}\right)<0.$$

Then the market maker would choose λ_n to be minus infinity, and there will be no equilibrium. (Note that the informed trader could deviate from his strategy by picking a different x_n which gives him a higher profit.) Second

$$\left(x_n+z_n\right)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L+\nu_n\right)-2\alpha_n\nu_n-d_{n+1}\gamma_{n+1}\right)>0$$

Then the market maker would choose λ_n to be plus infinity, and there will be no equilibrium. Third

$$\left(x_n+z_n\right)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L+\nu_n\right)-2\alpha_n\nu_n-d_{n+1}\gamma_{n+1}\right)=0.$$

Then the conditional expected profit of the market maker will be

$$\begin{aligned} &\left(p_{n-1}-\nu_n\right)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+\alpha_n\left(\sigma_n^2+\nu_n^2\right)-2p_{n-1}\alpha_n\nu_n \\ &+\alpha_n p_{n-1}^2+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\ = \;& \alpha_n\sigma_n^2+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \end{aligned}$$

independent of λ_n. Therefore the market maker has several strategies, he can pick whatever λ_n he desires. He actually participates here since

$$\begin{aligned} y_n &= -x_n-z_n-\alpha_n\left(\nu_{n-1}^L-p_n\right) \\ &= -d_{n+1}\gamma_{n+1}+\alpha_n\left(p_n-\nu_n\right). \end{aligned}$$

C Limit order traders

For each $n = 1, \ldots, N$ we assume that

$$E\left(\left(\nu - p_n\right)\alpha_n\left(\nu_{n-1}^L - p_n\right)\left|\mathcal{F}_{n-1}\right.\right) = 0$$

From this we get:

$$\begin{aligned}
& E\left(\left(\nu - p_n\right)\alpha_n\left(\nu_{n-1}^L - p_n\right)\left|\mathcal{F}_{n-1}\right.\right) = 0 \\
\Longleftrightarrow\quad & E\left(\alpha_n\left|\mathcal{F}_{n-1}\right.\right)\left(\left(\nu_{n-1}^L\right)^2 - 2p_{n-1}\nu_{n-1}^L + p_{n-1}^2\right) \\
& + E\left(\alpha_n\lambda_n\left(\alpha_n\right)\left|\mathcal{F}_{n-1}\right.\right)\gamma_n\left(-E\left(\nu^2\left|\mathcal{F}_{n-1}\right.\right) - \left(\nu_{n-1}^L\right)^2 + 4\nu_{n-1}^L p_{n-1} - 2p_{n-1}^2\right) \\
& + E\left(\alpha_n\lambda_n^2\left(\alpha_n\right)\left|\mathcal{F}_{n-1}\right.\right)\left(\gamma_n^2 E\left(\nu^2\left|\mathcal{F}_{n-1}\right.\right) + \gamma_n^2 p_{n-1}^2 - 2\gamma_n^2\nu_{n-1}^L p_{n-1} + \sigma_z^2\Delta t_n\right) \\
= \quad & 0
\end{aligned}$$

For $\alpha_n \in \mathcal{F}_{n-1}$ and λ_n a deterministic function of α_n, we get:

$$\begin{aligned}
& E\left(\left(\nu - p_n\right)\alpha_n\left(\nu_{n-1}^L - p_n\right)\left|\mathcal{F}_{n-1}\right.\right) = 0 \\
\Longleftrightarrow\quad & \left(\nu_{n-1}^L\right)^2 - 2p_{n-1}\nu_{n-1}^L + p_{n-1}^2 \\
& + \lambda_n\left(\alpha_n\right)\left(-E\left(\nu^2\left|\mathcal{F}_{n-1}\right.\right) - \left(\nu_{n-1}^L\right)^2 + 4\nu_{n-1}^L p_{n-1} - 2p_{n-1}^2\right) \\
& + \lambda_n^2\left(\alpha_n\right)\left(\gamma_n^2 E\left(\nu^2\left|\mathcal{F}_{n-1}\right.\right) + \gamma_n^2 p_{n-1}^2 - 2\gamma_n^2\nu_{n-1}^L p_{n-1} + \sigma_z^2\Delta t_n\right) = 0
\end{aligned}$$

D Calculation of c_n, d_n and e_n

The purpose of this section is to make a previous remark more precise. In the previous subsections we omitted the explicit calculation of the parameters c_n, d_n and e_n. Now we shall give explicit expressions for them. The calculations are quite lengthy and they are just stated here for the convenience of the reader. However they are necessary since they will provide us with the result that c_{n+1}, d_{n+1} and e_{n+1} are $\mathcal{M}_n$−measurable functions, a property which was used in the proofs given in the previous parts of this appendix.

$$\begin{aligned}
&\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M \,|\mathcal{M}_n\right)\\
= \; &\frac{\left(x_n + z_n + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - \nu_n\right) - d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}\\
&+ \left(p_{n-1} - \nu_n\right)\left(x_n + z_n + \alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) + c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2 + \nu_n^2\right)\\
&- 2p_{n-1}c_{n+1}\gamma_{n+1}^2\nu_n + c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}\nu_n - d_{n+1}\gamma_{n+1}p_{n-1}\\
&+ c_{n+1}\sigma_z^2\Delta t_{n+1} + e_{n+1}.
\end{aligned}$$

Now plugging in the expression for ν_n,

$$\begin{aligned}
\nu_n &= \nu_{n-1} + \frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(x_n + z_n - \gamma_n\left(\nu_{n-1} - p_{n-1}\right)\right)\\
&= \nu_{n-1} + g_n\left(x_n + z_n\right) - g_n\left(\nu_{n-1} - p_{n-1}\right) = h_n + g_n\left(x_n + z_n\right)
\end{aligned}$$

where

$$\begin{aligned}
&g_n := \frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}},\\
&h_n := \nu_{n-1} - g_n\gamma_n\left(\nu_{n-1} - p_{n-1}\right)\\
&s_n := 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - \left(h_n + g_n\left(x_n + z_n\right)\right)\right) - d_{n+1}\gamma_{n+1}\\
&r_n := \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - h_n\right) - d_{n+1}\gamma_{n+1}
\end{aligned}$$

one gets

$$\begin{aligned}
&\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M \,|\mathcal{M}_n\right)\\
= \; &\frac{\left(x_n + z_n + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n + g_n\left(x_n + z_n\right)\right) + s_n\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}\\
&+ \left(p_{n-1} - \left(h_n + g_n\left(x_n + z_n\right)\right)\right)\left(x_n + z_n + \alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right)\\
&+ c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2 + \left(h_n + g_n\left(x_n + z_n\right)\right)^2\right)\\
&- 2p_{n-1}c_{n+1}\gamma_{n+1}^2\left(h_n + g_n\left(x_n + z_n\right)\right) + c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}\left(h_n + g_n\left(x_n + z_n\right)\right)\\
&- d_{n+1}\gamma_{n+1}p_{n-1} + c_{n+1}\sigma_z^2\Delta t_{n+1} + e_{n+1}.
\end{aligned}$$

Simplifying leads to

$$
\begin{aligned}
&\max_{(\lambda_n,\dots,\lambda_N)} E\left(\pi_n^M \,|\mathcal{M}_n\right) \\
= \;& \frac{\left((x_n+z_n)\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right)+r_n\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
&+ \left(p_{n-1}-h_n\right)(x_n+z_n) + \left(p_{n-1}-h_n\right)\left(\alpha_n \nu_{n-1}^L - \alpha_n p_{n-1}\right) \\
&- g_n (x_n+z_n)(x_n+z_n) - g_n (x_n+z_n)\left(\alpha_n \nu_{n-1}^L - \alpha_n p_{n-1}\right) \\
&+ c_{n+1}\gamma_{n+1}^2\sigma_n^2 + c_{n+1}\gamma_{n+1}^2 h_n^2 + 2c_{n+1}\gamma_{n+1}^2 h_n g_n (x_n+z_n) \\
&+ c_{n+1}\gamma_{n+1}^2 g_n^2 (x_n+z_n)^2 - 2p_{n-1}c_{n+1}\gamma_{n+1}^2 h_n - 2p_{n-1}c_{n+1}\gamma_{n+1}^2 g_n (x_n+z_n) \\
&+ c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}h_n + d_{n+1}\gamma_{n+1} g_n (x_n+z_n) \\
&- d_{n+1}\gamma_{n+1}p_{n-1} + c_{n+1}\sigma_z^2 \Delta t_{n+1} + e_{n+1}.
\end{aligned}
$$

Factorizing $x_n + z_n$:

$$
\begin{aligned}
&\max_{(\lambda_n,\dots,\lambda_N)} E\left(\pi_n^M \,|\mathcal{M}_n\right) \\
= \;& \frac{\left((x_n+z_n)\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right)+r_n\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
&+ (x_n+z_n)\left(\left(p_{n-1}-h_n\right) - g_n\left(\alpha_n \nu_{n-1}^L - \alpha_n p_{n-1}\right)\right) \\
&+ (x_n+z_n)\left(2c_{n+1}\gamma_{n+1}^2 h_n g_n - 2p_{n-1}c_{n+1}\gamma_{n+1}^2 g_n + d_{n+1}\gamma_{n+1} g_n\right) \\
&+ (x_n+z_n)^2\left(-g_n + c_{n+1}\gamma_{n+1}^2 g_n^2\right) + \left(p_{n-1}-h_n\right)\left(\alpha_n \nu_{n-1}^L - \alpha_n p_{n-1}\right) \\
&+ c_{n+1}\gamma_{n+1}^2\sigma_n^2 + c_{n+1}\gamma_{n+1}^2 h_n^2 \\
&- 2p_{n-1}c_{n+1}\gamma_{n+1}^2 h_n + c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}h_n - d_{n+1}\gamma_{n+1}p_{n-1} \\
&+ c_{n+1}\sigma_z^2 \Delta t_{n+1} + e_{n+1} \\
= \;& (x_n+z_n)^2 \frac{\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
&+ (x_n+z_n)\frac{2\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right)}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
&\times \left(\alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right) - d_{n+1}\gamma_{n+1}\right) \\
&+ \frac{\left(\alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right) - d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
&+ (x_n+z_n)\left(p_{n-1}-h_n - g_n\left(\alpha_n \nu_{n-1}^L - \alpha_n p_{n-1}\right)\right) \\
&+ (x_n+z_n)\left(2c_{n+1}\gamma_{n+1}^2 h_n g_n - 2p_{n-1}c_{n+1}\gamma_{n+1}^2 g_n + d_{n+1}\gamma_{n+1} g_n\right) \\
&+ (x_n+z_n)^2\left(-g_n + c_{n+1}\gamma_{n+1}^2 g_n^2\right) + \left(p_{n-1}-h_n\right)\left(\alpha_n \nu_{n-1}^L - \alpha_n p_{n-1}\right) \\
&+ c_{n+1}\gamma_{n+1}^2\sigma_n^2 + c_{n+1}\gamma_{n+1}^2 h_n^2 \\
&- 2p_{n-1}c_{n+1}\gamma_{n+1}^2 h_n + c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}h_n \\
&- d_{n+1}\gamma_{n+1}p_{n-1} + c_{n+1}\sigma_z^2 \Delta t_{n+1} + e_{n+1}.
\end{aligned}
$$

We set

$$
c_n := \frac{\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} - g_n + c_{n+1}\gamma_{n+1}^2 g_n^2,
$$

$$
\begin{aligned}
d_n := &\frac{2\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right)}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
\times &\left(\alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right) - d_{n+1}\gamma_{n+1}\right) \\
+ &\; p_{n-1} - h_n - g_n\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) + 2c_{n+1}\gamma_{n+1}^2 h_n g_n - 2p_{n-1}c_{n+1}\gamma_{n+1}^2 g_n + d_{n+1}\gamma_{n+1}g_n
\end{aligned}
$$

and

$$
\begin{aligned}
e_n := &\frac{\left(\alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right) - d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
+ &\; \left(p_{n-1}-h_n\right)\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) + c_{n+1}\gamma_{n+1}^2\sigma_n^2 + c_{n+1}\gamma_{n+1}^2 h_n^2 \\
- &\; 2p_{n-1}c_{n+1}\gamma_{n+1}^2 h_n + c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}h_n \\
- &\; d_{n+1}\gamma_{n+1}p_{n-1} + c_{n+1}\sigma_z^2\Delta t_{n+1} + e_{n+1}.
\end{aligned}
$$

Resubstituting h_n and g_n, using

$$
h_n = \nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1} - p_{n-1}\right)
$$

yields the following expressions:

$$
\begin{aligned}
c_n \; = \; &\frac{\left(1+\alpha_n\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}} - 2c_{n+1}\gamma_{n+1}^2\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} - \frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}} \\
&+ c_{n+1}\gamma_{n+1}^2\left(\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
d_n = &\frac{2\left(1+\alpha_n\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}} - 2c_{n+1}\gamma_{n+1}^2\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
\times &\left(\alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)\right. \\
+ &\left.\; 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - \left(\nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)\right) - d_{n+1}\gamma_{n+1}\right) \\
+ &\; p_{n-1} - \left(\nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right) \\
- &\; \frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) \\
+ &\; 2c_{n+1}\gamma_{n+1}^2\left(\nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}} \\
- &\; 2p_{n-1}c_{n+1}\gamma_{n+1}^2\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}} + d_{n+1}\gamma_{n+1}\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
e_n = &\frac{1}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
\times &\left(\alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1} - p_{n-1}\right)\right)\right)^2 \\
+ &\frac{1}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
\times &\left(2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - \nu_{n-1} + \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1} - p_{n-1}\right)\right) - d_{n+1}\gamma_{n+1}\right)^2 \\
+ &\frac{\alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1} - p_{n-1}\right)\right)}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\
\times &\left(4c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - \nu_{n-1} + \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1} - p_{n-1}\right)\right) - 2d_{n+1}\gamma_{n+1}\right) \\
+ &\left(p_{n-1} - \left(\nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1} - p_{n-1}\right)\right)\right)\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) \\
+ &c_{n+1}\gamma_{n+1}^2\sigma_n^2 + c_{n+1}\gamma_{n+1}^2\left(\nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1} - p_{n-1}\right)\right)^2 \\
- &2p_{n-1}c_{n+1}\gamma_{n+1}^2\left(\nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1} - p_{n-1}\right)\right) + c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 \\
+ &d_{n+1}\gamma_{n+1}\left(\nu_{n-1} - \frac{\gamma_n^2\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1} - p_{n-1}\right)\right) \\
- &d_{n+1}\gamma_{n+1}p_{n-1} + c_{n+1}\sigma_z^2\Delta t_{n+1} + e_{n+1}.
\end{aligned}
$$

This shows that indeed, $c_{n+1}, d_{n+1}, e_{n+1}$ are $\mathcal{M}_n$−measurable. Obviously, here we used that $\gamma_{n+1} \in \mathcal{M}_n$. From here one can also easily find the formulas for c_n, d_n and e_n if the order book is predictable. We can further simplify the formula for c_n:

$$
\begin{aligned}
c_n \;=\; &\frac{\left(1 + \alpha_n\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}} - 2c_{n+1}\gamma_{n+1}^2\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} - \frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}} \\
&+ c_{n+1}\gamma_{n+1}^2\left(\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\right)^2.
\end{aligned}
$$

This leads to

$$\begin{aligned} c_n &= \frac{1}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} + \frac{\alpha_n^2\left(\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} + \frac{\alpha_n\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}}{2\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\ &+ \frac{c_{n+1}^2\gamma_{n+1}^4\left(\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)^2}{\alpha_n - c_{n+1}\gamma_{n+1}^2} - \frac{c_{n+1}\gamma_{n+1}^2\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}}{\alpha_n - c_{n+1}\gamma_{n+1}^2} \\ &- \frac{\alpha_n c_{n+1}\gamma_{n+1}^2\left(\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)^2}{\alpha_n - c_{n+1}\gamma_{n+1}^2} - \frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}} \\ &+ c_{n+1}\gamma_{n+1}^2\left(\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)^2 \end{aligned}$$

and from there

$$\begin{aligned} c_n &= \frac{1}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} + \frac{\alpha_n^2\left(\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} + \frac{\alpha_n\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}}{2\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} \\ &- \frac{c_{n+1}\gamma_{n+1}^2\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}}{\alpha_n - c_{n+1}\gamma_{n+1}^2} - \frac{2\alpha_n\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}}{2\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} + \frac{\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}c_{n+1}\gamma_{n+1}^2}{\alpha_n - c_{n+1}\gamma_{n+1}^2} \\ &= \frac{\left(1-\alpha_n\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} = \frac{\left(1-\alpha_n\gamma_n\frac{\sigma_n^2}{\sigma_z^2\Delta t_{n-1}}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}. \end{aligned}$$

Bibliography

[Asm98] S. Asmussen. Subexponential asymptotics for stochastic processes: Extremal behavior, stationary distributions and first passage probabilities. *The Annals of Applied Probability* **8 (2)**, 354–374 (1998).

[Bac92] K. Back. Insider trading in continuous time. *Review of Financial Studies* **5**, 387–409 (1992).

[BBS97] P. Bak, M. Baczuski, and M. Shubik. Price variations in a stock market with many agents. *Physica A* **246**, 430 (1997).

[BDW97] T. Bollerslev, I. Domowitz, and J. Wang. Order flow and the bid-ask spread: An empirical probability model of screen-based trading. *J. Econ. Dynamics Control* **21**, 1471 (1997).

[BHS95] P. Bias, P. Hilton, and C. Spatt. An empirical analysis of the limit order book and the order flow in the Paris bourse. *Journal of Finance* **50** (1995).

[BJ03] O. Bondarenko and S. Jaeyoung. Specialist participation and limit orders. *Journal of Financial Markets* **6**, 539–571 (2003).

[Bla95] F. Black. Equilibrium exchanges. *Financial Analysts Journal* **51**, 23 29 (1995).

[BMP02] J.-P. Bouchaud, M. Mezard, and M. Potters. Statistical properties of stock order books: Empirical results and models. *Quantitative Finance* **2 (4)**, 251–256 (2002).

[BMS02] M. Braverman, T. Mikosch, and G. Samorodnitsky. Tail probabilities of subadditive functionals of Lévy processes. *The Annals of Applied Probability* **12 (1)**, 69–100 (2002).

[Bon01] O. Bondarenko. Competing market makers, liquidity provision, and bid-ask spreads. *Journal of Financial Markets* **4**, 269–308 (2001).

[Bra05] M. Braverman. On a class of Lévy processes. *Statistics and Probability Letters* **75**, 179–189 (2005).

[BS91] U. Bhattacharya and M. Spiegel. Insiders, outsiders and market breakdowns. *Review of Financial Studies* **4**, 225–282 (1991).

[BS02] A. BORODIN AND P. SALMINEN. "Handbook of Brownian Motion: Facts and Formulae". Birkhäuser, Basel (2002).

[CC98] K.H. CHUNG AND C. CHAROENWONG. Insider trading and the bid-ask spread. *Financial Review* **33**, 1–20 (1998).

[CCM85] K.J. COHEN, R.M. CONROY, AND S.F. MAIER. "Order Flow and the Quality of the Market: Market Making and the Changing Structure of the Securities Industry, Edited by Amihud, Y., Ho, T. and Schwartz, R." Lexington Books, Lexington, MA (1985).

[CH95] S. CHAKRAVARTY AND C. HOLDEN. An integrated model of market and limit orders. *Journal of Financial Intermediation* **4**, 213–241 (1995).

[CHS94] W. CHRISTIE, J. HARRIS, AND P. SCHULTZ. Why did NASDAQ market makers stop avoiding odd-eight quotes? *Journal of Finance* **49**, 1841–1860 (1994).

[CI02] C. CHIARELLA AND G. IORI. A simulation analysis of the microstructure of double auction markets. *Quant. Finance* **2**, 346 (2002).

[CJP04] U. CETIN, R. JARROW, AND P. PROTTER. Liquidity risk and arbitrage pricing theory. *Finance and Stochastics* **8**, 311–341 (2004).

[CS94] W. CHRISTIE AND P. SCHULTZ. Why do NASDAQ market makers avoid odd-eight quotes? *Journal of Finance* **49**, 1813–1840 (1994).

[CS01] D. CHALLET AND R. STINCHCOMBE. Analyzing and modelling 1+1d markets. *Physica A* **300**, 285 (2001).

[CVNVN99] K. CHUNG, B. VAN NESS, AND R. VAN NESS. Limit orders and the bid-ask spread. *Journal of Financial Economics* **53**, 255–287 (1999).

[Den93] J. DENNERT. Price competition between market makers. *Review of Economic Studies* **60**, 735–751 (1993).

[DFIS03] M.G. DANIELS, J.D. FARMER, G. IORI, AND E. SMITH. Quantitative model of price diffusion and market friction based on trading as a mechanistic random process. *Phys. Rev. Lett.* **90** (2003).

[DM97] P. DUTTA AND A. MADHAVAN. Competition and collusion in dealer markets. *Journal of Finance* **52**, 245–276 (1997).

[DS95] F. DELBAEN AND W. SCHACHERMAYER. Arbitrage possibilities in Bessel processes and their relations to local martingales. *Probab. Theory Related Fields* **102**, 357–366 (1995).

[DS98] F. DELBAEN AND W. SCHACHERMAYER. The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen* **312**, 215–250 (1998).

[DW94] I. DOMOWITZ AND J. WANG. Auctions as algorithms. *J. Econ. Dynamics Control* **18**, 29 (1994).

[EI98] D. ELIEZER AND KOGAN I. Scaling laws for the market microstructure of the interdealer broker markets. *Preprint cond.mat/9808240* (1998).

[Fer99] R. FERNHOLZ. On the diversity of equity markets. *Journal of Mathematical Economics* **31**, 393–417 (1999).

[Fer01] R. FERNHOLZ. Equity portfolios generated by ranked market weights. *Finance and Stochastics* **5**, 469–486 (2001).

[Fer02] R. FERNHOLZ. "Stochastic Portfolio Theory". Springer Verlag, New York (2002).

[FK97] H. FÖLLMER AND D. KRAMKOV. Optional decompositions under constraints. *Probab. Theory Relat. Fields* **109**, 1–25 (1997).

[FKK05] R. FERNHOLZ, I. KARATZAS, AND C. KARDARAS. Diversity and relative arbitrage in equity markets. *Finance and Stochastics* **9**, 1–27 (2005).

[GD92] R. GOSSEN-DOMBROWSKY. Ein Ansatz zur Optionsbewertung in stochastischen Modellen ohne ein äquivalentes Martingalmaß. Master's thesis, Bonn University (1992).

[Glo89] L. GLOSTEN. Insider trading, liquidity and the role of the monopolist specialist. *Journal of Business* **62**, 211–235 (1989).

[GPGS00] P. GOPIKRISHNAN, V. PLEROU, X. GABAIX, AND H. STANLEY. Statistical properties of share volume traded in financial markets. *Physical Review E* **62(4)**, R4493–R4496 (2000).

[HH96a] L. HARRIS AND J. HASBROUCK. Market versus limit orders: The SuperDOT evidence on order submission strategy. *Journal of Financial and Quantitative Analysis* **31(2)**, 213–231 (1996).

[HH96b] L. HARRIS AND J. HASBROUCK. Market versus limit orders: The SuperDOT evidence on order submission strategy. *Journal of Financial and Quantitative Analysis* **31**, 213–231 (1996).

[HL05a] H. HULT AND F. LINDSKOG. Extremal behavior of regularly varying stochastic processes. *Stochastic Processes and their Applications* **115**, 249–274 (2005).

[HL05b] H. HULT AND F. LINDSKOG. Extremal behavior of stochastic integrals driven by regularly varying processes. *To appear in The Annals of Probability* (2005).

[HLMS05] H. HULT, F. LINDSKOG, T. MIKOSCH, AND G. SAMORODNITSKY. Functional large deviations for multivariate regularly varying random walks. *The Annals of Applied Probability* **15 (4)**, 2651–2680 (2005).

[HMS04] P. HOLLIFIELD, R. MILLER, AND P. SANDAS. Empirical analysis of limit order markets. *Review of Economic Studies* **71 (4)**, 1027–1063 (2004).

[HS93] J. HASBROUCK AND G. SOFIANOS. The trades of market makers: An empirical analysis of NYSE specialists. *Journal of Finance* **48**, 1565–1593 (1993).

[Jac79] J. JACOD. "Calcul Stochastiques et Problèmes de Martingales. (Lecture Notes in Mathematics, Vol. 714)". Springer, Berlin Heidelberg New York (1979).

[JGL94] C. JONES, K. GAUTAM, AND M. LIPSON. Information, trading and volatility. *Journal of Financial Economics* **36(1)**, 127–154 (1994).

[JKL94] C. JONES, G. KAUL, AND M. LIPSON. Transactions, volume and volatility. *The Review of Financial Studies* **7(4)**, 631–651 (1994).

[KS91] I. KARATZAS AND S.E. SHREVE. "Brownian Motion and Stochastic Calculus". Springer Verlag, New York (1991).

[Kyl85] A. KYLE. Continuous auctions and insider trading. *Econometrica* **53**, 1315–1335 (1985).

[Kyl89] A. KYLE. Informed speculation with imperfect competition. *The Review of Economic Studies* **56**, 317–356 (1989).

[Len77] E. LENGLART. Transformation des martingales locales par changement absolument continu de probabilités. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **39**, 65–70 (1977).

[LMZ02] A.W. LO, A.C. MACKINLAY, AND J. ZHANG. Econometric models of limit-order executions. *Journal of Financial Economics* **65**, 31–71 (2002).

[LZ03] P. LYNCH AND G. ZUMBACH. Market heterogeneities and the causal structure of volatility. *Quantitative Finance* **3**, 320–331 (2003).

[Mas00] S. MASLOV. Simple model of a limit order-driven market. *Physica A* **278**, 571 (2000).

[Men82] H. MENDELSON. Market behavior in a clearing house. *Econometrica* **50**, 1505 (1982).

[MM01] S. MASLOV AND M. MILLS. Price fluctuations from the order book perspective - empirical facts and a simple model. *Physica A* **299**, 234 (2001).

[MS98] A. MADHAVAN AND G. SOFIANOS. An empirical analysis of NYSE specialist trading. *Journal of Financial Economics* **48**, 189–210 (1998).

[OR06] J. OSTERRIEDER AND T. RHEINLÄNDER. Arbitrage opportunities in diverse markets via a non-equivalent measure change. *Annals of Finance* **2**, 287–301 (2006).

[PB03a] M. POTTERS AND J.-P. BOUCHAUD. More statistical properties of order books and price impact. *Physica A* **324**, 133–140 (2003).

[PB03b] M. POTTERS AND J.-P. BOUCHAUD. Statistical properties of stock order books. *Physica A* **324**, 133 (2003).

[PGG+01] V. Plerou, P. Gopikrishnan, X. Gabaix, L.A.N. Amaral, and H.E. Stanley. Price fluctuations, market activity and trading volume. *Quantitative Finance* **1(2)**, 262–269 (2001).

[PGS05] V. Plerou, P. Gopikrishnan, and H.E. Stanley. Quantifying fluctuations in market liquidity: Analysis of the bid-ask spread. *Physical Review E* **71**, 046131–1 – 046131–1 (2005).

[RS93] J. Rosinsky and G. Samorodnitsky. Distributions of subadditive functionals of sample paths of infinitely divisible processes. *The Annals of Probability* **21 (2)**, 996–1014 (1993).

[RV94] J.C. Rochet and J.L. Vila. Insider trading without normality. *The Review of Economic Studies* **61**, 131–152 (1994).

[RY04] D. Revuz and M. Yor. "Continuous Martingales and Brownian Motion". Springer Verlag, Berlin (2004).

[SFGK03] E. Smith, J.D. Farmer, L. Gillemot, and S. Krishnamurty. Statistical theory of the continuous double auction. *Quantitative Finance* **3**, 481–514 (2003).

[Sla01] F. Slanina. Mean-field approximation for a limit order driven market model. *Phys. Rev. E* **64**, 056136 (2001).

[Sof95] G. Sofianos. Specialist gross trading revenues at the New York stock exchange. *NYSE working paper* **Nr. 95-01** (1995).

[Tij03] H.C. Tijms. "A First Course in Stochastic Models". John Wiley and Sons Ltd., West Sussex, England (2003).

[TT99] L.H. Tang and G.S. Tian. Reaction-diffusion-branching models of stock price fluctuations. *Physica A* **264**, 543 (1999).

[Wil87] E. Willekens. On the supremum of an infinitely divisible process. *Stochastic processes and their applications* **26**, 173–175 (1987).

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