Order
https://doi.org/10.1007/s11083-022-09597-4

On Nontrivial Weak Dicomplementations and the Lattice Congruences that Preserve Them

Leonard Kwuida[1] · **Claudia Mureşan**[2]

Received: 4 February 2021 / Accepted: 25 February 2022

Abstract
We study the existence of nontrivial and of representable (dual) weak complementations, along with the lattice congruences that preserve them, in different constructions of bounded lattices, then use this study to determine the finite (dual) weakly complemented lattices with the largest numbers of congruences, along with the structures of their congruence lattices. It turns out that, if $n \geq 7$ is a natural number, then the four largest numbers of congruences of the n–element (dual) weakly complemented lattices are: $2^{n-2}+1$, $2^{n-3}+1$, $5 \cdot 2^{n-6}+1$ and $2^{n-4}+1$, which yields the fact that, for any $n \geq 5$, the largest and second largest numbers of congruences of the n–element weakly dicomplemented lattices are $2^{n-3}+1$ and $2^{n-4}+1$. For smaller numbers of elements, several intermediate numbers of congruences appear between the elements of these sequences.

Keywords (principal) congruence · (co)atom of a bounded lattice · (glued · ordinal · horizontal) sum of bounded lattices · (nontrivial · representable) (dual) weak complementation

Mathematics Subject Classification Primary 06B10 · Secondary 06F99

1 Introduction

Weakly dicomplemented lattices arise as abstractions of concept algebras, introduced by Rudolf Wille when modelling negation on concept lattices [24]; they are bounded lattices endowed with two unary operations, called *weak complementation* and *dual weak complementation*, together forming the *weak dicomplementation*, which generalize Boolean algebras; in fact if $(L, \wedge, \vee, ^-, 0, 1)$ is a Boolean algebra, then $(L, \wedge, \vee, ^-, ^-, 0, 1)$ is a weakly dicomplemented lattice. Their bounded lattice reducts endowed with the (dual) weak complementation are called **(dual) weakly complemented lattices**. Any bounded lattice can be

---

✉ Leonard Kwuida
leonard.kwuida@bfh.ch; kwuida@gmail.com

Claudia Mureşan
c.muresan@yahoo.com; cmuresan@fmi.unibuc.ro

1 Bern University of Applied Sciences, Bern, Switzerland

2 University of Bucharest, Bucharest, Romania

Published online: 29 November 2022

endowed with the *trivial weak dicomplementation*, formed of the weak complementation that sends 1 to 0 and all other elements to 1 and the dual weak complementation that sends 0 to 1 and all other elements to 0. If X is a set and c a closure operator on $\mathcal{P}(X)$, then the set of closed subsets of X forms a lattice and the operation $cA \mapsto c(X \setminus cA)$ is a weak complementation on the lattice of c-closed subsets of X. Dually, if k is a kernel operator on $\mathcal{P}(X)$, then $kB \mapsto k(X \setminus kB)$ is a dual weak complementation. Additional examples can be found in [16].

We take a purely lattice–theoretical approach to the study of these algebras and investigate the existence of nontrivial weak dicomplementations on glued and horizontal sums of bounded lattices, as well as (co)atomic bounded lattices with different numbers of (co) atoms and determine the lattice congruences that preserve those (dual) weak complementations. This allows us to determine all weak dicomplementations that can be defined on bounded lattices with certain lattice structures. Since the notion of representability is important in the study of these algebras, we also determine which of those (dual) weak complementations are representable.

After this preliminary investigation, we are able to determine the structures of the finite (dual) weakly complemented lattices, as well as weakly dicomplemented lattices, that have the largest numbers of congruences out of the algebras of the same kind with the same numbers of elements, along with the structures of their congruence lattices; we do this for weakly complemented lattices in our main theorem: Theorem 7.2, which also yields the dual results for dual weakly complemented lattices, and for weakly dicomplemented lattices in Corollary 7.1. This problem, which is also related to that of the representability of lattices as congruence lattices of various kinds of algebras, has been investigated for lattices in [5] and later in [3, 19], for semilattices in [4] and for bounded involution lattices, pseudo–Kleene algebras and antiortholattices in [20]; its counterpart for infinite algebras has been studied in [17, 18] and later in [2, 21].

2 Preliminaries

We will designate all algebras by their underlying sets. Recall that a variety $\mathbb{V}$ of similar algebras is said to be *semi–degenerate* iff all subalgebras of any nonsingleton member of $\mathbb{V}$ are nonsingleton. Clearly, any variety of bounded lattice–ordered algebras is semi–degenerate. We denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. $\dot{\cup}$ will be the disjoint union. For any set M, $|M|$ denotes the cardinality of M and $\mathcal{P}(M)$ denotes the power set of M; $\mathrm{Part}(M)$ and $(\mathrm{Eq}(M), \cap, \vee, =_M, M^2)$ will be the complete lattices of the partitions and the equivalences on M, respectively, where $\mathrm{Eq}(M)$ is ordered by the set inclusion, while the order $\leq$ of $\mathrm{Part}(M)$ is given by: for any $\pi, \rho \in \mathrm{Part}(M)$, $\pi \leq \rho$ iff every class of ρ is a union of classes from π, and $\mathrm{eq} : \mathrm{Part}(M) \to \mathrm{Eq}(M)$ will be the canonical lattice isomorphism. If $\{M_1, \dots, M_n\} \in \mathrm{Part}(M)$ for some $n \in \mathbb{N}^*$, then $\mathrm{eq}(\{M_1, \dots, M_n\})$ will be streamlined to $\mathrm{eq}(M_1, \dots, M_n)$. We will use Grätzer's notation for the lattice operations [11].

Let $\mathbb{V}$ be a variety of algebras of a similarity type τ, $\mathbb{C}$ a class of algebras with reducts in $\mathbb{V}$ and A and B algebras with reducts in $\mathbb{V}$. Then $\mathrm{S}_{\mathbb{V}}(\mathbb{C})$ will denote the class of the subalgebras of the τ–reducts of the members of $\mathbb{C}$, and $\mathrm{S}_{\mathbb{V}}(\{A\})$ will simply be denoted $\mathrm{S}_{\mathbb{V}}(A)$. We will abbreviate by $A \cong_{\mathbb{V}} B$ the fact that the τ–reducts of A and B are isomorphic.

$(\mathrm{Con}_{\mathbb{V}}(A), \cap, \vee, =_A, A^2)$ will be the complete lattice of the congruences of the τ–reduct of A. For any $n \in \mathbb{N}^*$ and any constants $\kappa_1, \dots, \kappa_n$ from τ, we denote by $\mathrm{Con}_{\mathbb{V}\kappa_1 \dots \kappa_n}(A) = \{\theta \in \mathrm{Con}_{\mathbb{V}}(A) \mid (\forall\, i \in [1, n])\, (\kappa_i^A/\theta = \{\kappa_i^A\})\}$, and, by extension, for any elements $a_1, \dots, a_n \in A$, by $\mathrm{Con}_{\mathbb{V}a_1 \dots a_n}(A) = \{\theta \in \mathrm{Con}_{\mathbb{V}}(A) \mid (\forall\, i \in [1,n])\,(a_i/\theta = \{a_i\})\}$, which is a complete sublattice of $\mathrm{Con}_{\mathbb{V}}(A)$ and thus a bounded lattice, according to the straightforward consequence [9, Lemma 2.(iii)] of [12, Corollary 2, p.51]. For any $X \subseteq A^2$ and any $a, b \in A$, we denote by $Cg_{\mathbb{V},A}(X)$ the congruence of the τ–reduct of A generated by X and, for brevity, the principal congruence $Cg_{\mathbb{V},A}(\{(a,b)\})$ by $Cg_{\mathbb{V},A}(a,b)$.

If $\mathbb{V}$ is the variety of (bounded) lattices, then the index $\mathbb{V}$ will be eliminated from the notations above.

For any poset $(P, \leq)$, $\mathrm{Min}(P)$ and $\mathrm{Max}(P)$ will be the set of the minimal elements and that of the maximal elements of $(P, \leq)$, respectively.

For any (bounded) lattice L, $\prec$ will denote the cover relation of L, L^d will be the dual of L and, if L has a 0, then the set of the atoms of L will be denoted by $\mathrm{At}(L)$, while, if L has a 1, then the set of the coatoms of L will be denoted by $\mathrm{CoAt}(L)$. Recall that a lattice with a 0 is said to be *atomic* iff each of its nonzero elements is lower bounded by an atom; dually for *coatomic* lattices. For any $a, b \in L$, we denote by $[a, b]_L = [a)_L \cap (b]_L$ the interval of L bounded by a and b; we eliminate the index L from this notation if L is $\mathbb{N}$ endowed with the natural order. Also, we denote by $a \| b$ the fact that a and b are incomparable. Note that all classes of any complete lattice congruence of a complete lattice, in particular all classes of any lattice congruence of a finite lattice, are intervals. $\mathrm{Ji}(L)$, $\mathrm{Sji}(L)$, $\mathrm{Mi}(L)$ and $\mathrm{Smi}(L)$ will be the sets of the join–irreducible, strictly (i.e. completely) join–irreducible, meet–irreducible and strictly (i.e. completely) meet–irreducible elements of L, respectively. For any $a \in \mathrm{Smi}(L)$, we will denote the unique cover of a in L by a^+, or by a^{+L} if the lattice L needs to be specified; similarly, for any $b \in \mathrm{Sji}(L)$, we denote by b^- or b^{-L} the unique lower cover of b in L. For all $n \in \mathbb{N}^*$, we denote by $\mathcal{C}_n$ the n–element chain.

Let L be a lattice with top element 1^L and M be a lattice with bottom element 0^M. Recall that the *glued sum* of L with M, denoted by $L \oplus M$, is the lattice obtained by stacking M on top of L and gluing the top element of L together with the bottom element of M; for the rigorous definition, see [9, 18], where we called this construction *ordinal sum*. For any $\alpha \in \mathrm{Con}(L)$ and any $\beta \in \mathrm{Con}(M)$, we denote by $\alpha \oplus \beta = Cg_{L \oplus M}(\alpha \cup \beta)$; clearly, $(L \oplus M)/(\alpha \oplus \beta) = (L/\alpha \setminus \{1^L/\alpha\}) \cup (M/\beta \setminus \{0^M/\beta\}) \cup \{1^L/\alpha \cup 0^M/\beta\}$ and the map $(\alpha, \beta) \mapsto \alpha \oplus \beta$ is a lattice isomorphism from $\mathrm{Con}(L) \times \mathrm{Con}(M)$ ($\cong \mathrm{Con}(L \times M)$) to $\mathrm{Con}(L \oplus M)$. Of course, the operation $\oplus$ on bounded lattices is associative, and so is the operation $\oplus$ on the congruences of such lattices.

Now let L and M be nonsingleton bounded lattices. Recall that the *horizontal sum* of L with M, denoted by $L \boxplus M$, is the nonsingleton bounded lattice obtained by gluing the bottom elements of L and M together, gluing their top elements together and letting all other elements of L be incomparable to every other element of M; for the rigorous definition see [9, 18]. For any $\alpha \in \mathrm{Con}(L)$ and any $\beta \in \mathrm{Con}(M)$, we denote by $\alpha \boxplus \beta = Cg_{L \boxplus M}(\alpha \cup \beta)$; clearly, if $\alpha \in \mathrm{Con}_1(L)$ and $\beta \in \mathrm{Con}_0(M)$, then $(L \boxplus M)/(\alpha \boxplus \beta) = (L \setminus \{1\})/\alpha \cup (M \setminus \{0\})/\beta \in \mathrm{Con}(L \boxplus M) \setminus \{(L \boxplus M)^2\}$. Note that the horizontal sum of nonsingleton bounded lattices is commutative and associative, it has $\mathcal{C}_2$ as a neutral element and it can be generalized to arbitrary families of nonsingleton bounded lattices; the operation $\boxplus$ on lattice congruences of those bounded lattices is commutative and associative, as well. The five–element modular non–distributive lattice is $\mathcal{M}_3 = \mathcal{C}_3 \boxplus \mathcal{C}_3 \boxplus \mathcal{C}_3$ and the five–element non–modular lattice is $\mathcal{N}_5 = \mathcal{C}_3 \boxplus \mathcal{C}_4$. For any nonzero cardinality κ, the modular lattice $\mathcal{M}_\kappa$ of length 3 with κ atoms is the horizontal sum of κ copies of the three–element chain.

3 The Algebras We Are Working With

Definition 3.1 Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $^\Delta$, $^\nabla$ be unary operations on L.

The algebra $(L, \wedge, \vee, ^\Delta, 0, 1)$ is called a *weakly complemented lattice* iff the unary operation $^\Delta$ is order–reversing and, for all $x, y \in L$, $x^{\Delta\Delta} \leq x$ and $(x \wedge y) \vee (x \wedge y^\Delta) = x$. In this case, the operation $^\Delta$ is called *weak complementation* on the bounded lattice L.

The algebra $(L, \wedge, \vee, ^\nabla, 0, 1)$ is called a *dual weakly complemented lattice* iff the unary operation $^\nabla$ is order–reversing and, for all $x, y \in L$, $x \leq x^{\nabla\nabla}$ and $(x \vee y) \wedge (x \vee y^\nabla) = x$. In this case, the operation $^\nabla$ is called *dual weak complementation* on L.

The algebra $(L, \wedge, \vee, ^\Delta, ^\nabla, 0, 1)$ is called a *weakly dicomplemented lattice* iff $(L, \wedge, \vee, ^\Delta, 0, 1)$ is a weakly complemented lattice and $(L, \wedge, \vee, ^\nabla, 0, 1)$ is a dual weakly complemented lattice. In this case, the pair $(^\Delta, ^\nabla)$ is called *weak dicomplementation* on L.

If A is a bounded lattice, $^\Delta$ is a weak complementation and $^\nabla$ is a dual weak complementation on A, then the abbreviated notations $(A, ^\Delta)$, $(A, ^\nabla)$ and $(A, ^\Delta, ^\nabla)$ will designate the weakly complemented lattice $(A, \wedge, \vee, ^\Delta, 0, 1)$, the dual weakly complemented lattice $(A, \wedge, \vee, ^\nabla, 0, 1)$, and the weakly dicomplemented lattice $(A, \wedge, \vee, ^\Delta, ^\nabla, 0, 1)$, respectively.

Note that the denomination of weak complementation is also used for the notion of semicomplementation in lattices with smallest element [1, 22].

We denote by $\mathbb{BA}$, $\mathbb{WCL}$, $\mathbb{DWCL}$ and $\mathbb{WDL}$ the varieties of Boolean algebras, weakly complemented lattices, dual weakly complemented lattices and weakly dicomplemented lattices, respectively.

It is immediate that any $L \in \mathbb{WCL}$ satisfies the identities: $0^\Delta \approx 1$, $1^\Delta \approx 0$, $x \vee x^\Delta \approx 1$ and $(x \wedge y)^\Delta \approx x^\Delta \vee y^\Delta$, as well as the quasiequations: $x^\Delta \approx 0 \to x \approx 1$, $x \wedge y \approx 0 \to x^\Delta \geq y$ and $x^\Delta \leq y \to y^\Delta \leq x$.

Dually, any $L \in \mathbb{DWCL}$ satisfies the identities: $0^\nabla \approx 1$, $1^\nabla \approx 0$, $x \wedge x^\nabla \approx 0$ and $(x \vee y)^\nabla \approx x^\nabla \wedge y^\nabla$, as well as the quasiequations: $x^\nabla \approx 1 \to x \approx 0$, $x \vee y \approx 1 \to x^\nabla \leq y$ and $x^\nabla \geq y \to y^\nabla \geq x$.

Additionally, any $L \in \mathbb{WDL}$ satisfies: $x^\nabla \leq x^\Delta$.

Clearly, $\mathbb{BA} \subseteq \mathbb{WCL} \cap \mathbb{DWCL}$, because the Boolean complementation of any Boolean algebra A is a weak complementation, as well as a dual weak complementation on A. Hence $\mathbb{BA}$ can be considered as a subvariety of $\mathbb{WDL}$ with an extended signature, by endowing each Boolean algebra with a second unary operation equalling its Boolean complementation. Note that, w.r.t. this weak dicomplementation, $\mathrm{Con}_{\mathbb{WDL}}(A) = \mathrm{Con}_{\mathbb{WCL}}(A) = \mathrm{Con}_{\mathbb{DWCL}}(A) = \mathrm{Con}_{\mathbb{BA}}(A) = \mathrm{Con}(A)$. Moreover, from the above it is easy to notice that, in a weakly dicomplemented lattice L, the weak complementation coincides with the weak dicomplementation iff L is a Boolean algebra and each of these operations coincides with the Boolean complementation of L. Hence $\mathbb{BA}$ with the extended signature is exactly the subvariety of $\mathbb{WDL}$ axiomatized by $x^\Delta \approx x^\nabla$.

Let us notice that any bounded lattice L can be organized as a weakly complemented lattice by endowing it with the *trivial weak complementation*: $x^\Delta = 1$ for all $x \in L \setminus \{1\}$, and it can be organized as a dual weakly complemented lattice by endowing it with the *trivial dual weak complementation*: $x^\nabla = 0$ for all $x \in L \setminus \{0\}$, hence it can be organized as a weakly dicomplemented lattice by endowing it with the *trivial weak dicomplementation*: $(^\Delta, ^\nabla)$, where $^\Delta$ is the trivial weak complementation and $^\nabla$ is the trivial dual weak complementation on L.

Since $\mathbb{WDL} \models \{x \wedge y \approx 0 \rightarrow x^{\Delta} \geq y, x \vee y \approx 1 \rightarrow x^{\nabla} \leq y\}$, it clearly follows that, for any cardinality $\kappa \geq 3$, the bounded lattice $\mathcal{M}_{\kappa}$ can only be endowed with the trivial weak dicomplementation.

Of course, for any $L \in \mathbb{WDL}$, if we consider its reducts from $\mathbb{WCL}$ and $\mathbb{DWCL}$, then $\mathrm{Con}_{\mathbb{WDL}}(L) = \mathrm{Con}_{\mathbb{WCL}}(L) \cap \mathrm{Con}_{\mathbb{DWCL}}(L)$.

Clearly, the trivial weak complementation is the (pointwise) largest weak complementation on L, while the trivial dual weak complementation is the (pointwise) smallest dual weak complementation on L. If $(^{\Delta 1}, ^{\nabla 1})$ and $(^{\Delta 2}, ^{\nabla 2})$ are weak dicomplementations on a bounded lattice L, then we say that $(^{\Delta 1}, ^{\nabla 1})$ is *smaller* than $(^{\Delta 2}, ^{\nabla 2})$ iff $^{\Delta 1}$ is pointwise smaller than $^{\Delta 2}$ and $^{\nabla 1}$ is pointwise larger than $^{\nabla 2}$. According to this definition, the trivial weak dicomplementation is the largest weak dicomplementation on any bounded lattice.

As mentioned in Section 1, the basic example of a weakly dicomplemented lattice is the canonical concept algebra associated to a context. A *context* is a triple (G, M, I), where G and M are sets and $I \subseteq G \times M$ is a binary relation; the elements of G are called *objects*, and elements of M are called *attributes*. For every $A \subseteq G$ and every $B \subseteq M$, we denote by: $\begin{cases} A' = \{m \in M \mid (\forall\, a \in A)\,(aIm)\}, \\ B' = \{g \in G \mid (\forall\, b \in B)\,(gIb)\}. \end{cases}$ The operations $' : \mathcal{P}(G) \rightarrow \mathcal{P}(M)$ and $' : \mathcal{P}(M) \rightarrow \mathcal{P}(G)$ are called *derivation of objects* and *of attributes*, respectively. The *concept algebra associated to the context (G, M, I)* is the weakly dicomplemented lattice $(\mathcal{B}(G, M, I), \wedge, \vee, ^{\Delta}, ^{\nabla}, 0, 1)$, where: $(\mathcal{B}(G, M, I) = \{(A, B) \mid A \subseteq G, B \subseteq M, A' = B, B' = A\}, \wedge, \vee)$ is a lattice whose order $\leq$ is defined by $(A, B) \leq (C, D)$ iff $A \subseteq C$ and $B \supseteq D$ for any $(A, B), (C, D) \in \mathcal{B}(G, M, I)$, with first element $0 = (\emptyset'', M)$ and last element $1 = (G, G')$, endowed with the weak complementation defined by $(A, B)^{\Delta} = ((G \setminus A)'', (G \setminus A)')$ and the dual weak complementation defined by $(A, B)^{\nabla} = ((M \setminus B)', (M \setminus B)'')$ for all $(A, B) \in \mathcal{B}(G, M, I)$.

Whenever J is a join–dense subset and M is a meet–dense subset of a complete lattice L, we have $L \cong \mathcal{B}(J, M, \leq)$, because the map $\varphi_{L,J,M} : L \rightarrow \mathcal{B}(J, M, \leq)$, defined by $\varphi_{L,J,M}(x) = (J \cap (x]_L, M \cap [x)_L)$ for all $x \in L$, is a lattice isomorphism. In this case, L can be endowed with the weak dicomplementation $(^{\Delta J}, ^{\nabla M})$ defined by $x^{\Delta J} = \bigvee (J \setminus (x]_L)$ and $x^{\nabla M} = \bigwedge (M \setminus [x)_L)$ for all $x \in L$. Note that the subsets $J \cup \{0\}$ and $J \setminus \{0\}$ of L are also join–dense, the subsets $M \cup \{1\}$ and $M \setminus \{1\}$ of L are also meet–dense, $^{\Delta J} = ^{\Delta(J \cup \{0\})} = ^{\Delta(J \setminus \{0\})}$, $^{\nabla M} = ^{\nabla(M \cup \{1\})} = ^{\nabla(M \setminus \{1\})}$ and, for every $H \in \{J, J \cup \{0\}, J \setminus \{0\}\}$ and every $N \in \{M, M \cup \{1\}, M \setminus \{1\}\}$, $\varphi_{L,J,M} = \varphi_{L,H,N}$, thus, furthermore, since this map is a weakly dicomplemented lattice isomorphism, the canonical weakly dicomplemented lattices $\mathcal{B}(J, M, \leq)$ and $\mathcal{B}(H, N, \leq)$ coincide. We say that a weak complementation $^{\Delta}$, respectively a dual weak complementation $^{\nabla}$ on L is *representable* iff $^{\Delta} = ^{\Delta J}$ for some join–dense subset J of L, respectively $^{\nabla} = ^{\nabla M}$ for some meet–dense subset M of L; we say that a weak dicomplementation $(^{\Delta}, ^{\nabla})$ on L is *representable* iff $^{\Delta}$ and $^{\nabla}$ are representable. Clearly, the trivial weak dicomplementation on L is representable, since it equals $(^{\Delta L}, ^{\nabla L})$. By [7, Theorem 4.(*ii*)], all weak dicomplementations on a finite distributive lattice are representable; it is not known whether this property still holds without distributivity, but it fails without finiteness: by [16, Theorem 4], the Boolean complementation of a complete atomfree Boolean algebra B is not a representable weak complementation or a representable dual weak complementation on B; for instance, if T is an infinite set and F is the Boolean filter of $\mathcal{P}(T)$ consisting of the cofinite subsets of T, then $\mathcal{P}(T)/F$ is a complete Boolean algebra with no join–irreducibles and thus no atoms [23, p.16].

In particular, if L is a complete dually algebraic lattice, then $\mathrm{Sji}(L)$ is join–dense in L [8, Theorem I.4.25], [10, Lemma 1.3.2], thus so is $\mathrm{Ji}(L)$, hence $L \cong \mathcal{B}(\mathrm{Sji}(L), L, \leq) \cong \mathcal{B}(\mathrm{Ji}(L), L, \leq)$, so that L can be endowed with the weak complementation ${}^{\Delta \mathrm{Ji}(L)}$, as well as the weak complementation ${}^{\Delta \mathrm{Sji}(L)}$. Dually, if L is an algebraic lattice, then $\mathrm{Smi}(L)$ is meet–dense in L, thus so is $\mathrm{Mi}(L)$, hence $L \cong \mathcal{B}(L, \mathrm{Smi}(L), \leq) \cong \mathcal{B}(L, \mathrm{Mi}(L), \leq)$, which can be endowed with the dual weak complementations ${}^{\nabla \mathrm{Mi}(L)}$ and ${}^{\nabla \mathrm{Smi}(L)}$.

If a lattice L is complete, algebraic and dually algebraic, then $\mathrm{Sji}(L)$ is join–dense in L and $\mathrm{Smi}(L)$ is meet–dense in L, therefore $L \cong \mathcal{B}(\mathrm{Sji}(L), \mathrm{Smi}(L), \leq)$, that can be endowed with the weak dicomplementation $({}^{\Delta \mathrm{Sji}(L)}, {}^{\nabla \mathrm{Smi}(L)})$, which, according to [7, p.236], is the smallest weak dicomplementation on L. Consequently, L has nontrivial weak complementations iff ${}^{\Delta \mathrm{Sji}(L)}$ is nontrivial, and L has nontrivial dual weak complementations iff ${}^{\nabla \mathrm{Smi}(L)}$ is nontrivial.

In particular, if L is a finite lattice, then $L \cong \mathcal{B}(\mathrm{Ji}(L), \mathrm{Mi}(L), \leq)$ and the smallest weak dicomplementation on L is $({}^{\Delta \mathrm{Ji}(L)}, {}^{\nabla \mathrm{Mi}(L)})$, so that L has nontrivial weak complementations iff ${}^{\Delta \mathrm{Ji}(L)}$ is nontrivial, and L has nontrivial dual weak complementations iff ${}^{\nabla \mathrm{Mi}(L)}$ is nontrivial.

Now let L be a bounded lattice and J, M subsets of L. We consider the following condition:

$$\neg sg\Delta(L, J, M): \quad (\exists\, m, n \in M)\, (J \subseteq (m]_L \cup (n]_L)$$

Clearly, if $\bigvee J$ exists in L, in particular if J is finite, then condition $\neg sg\Delta(L, J, M)$ implies: $(\exists\, m, n \in M)\, (\bigvee J \in (m \vee n]_L)$. Hence, if, furthermore, $1 \notin M$ and $\bigvee J = 1$, in particular if J is join–dense in L, then $\neg sg\Delta(L, J, M)$ is equivalent to:

$$(\exists\, m, n \in M)\, (m \neq n \textit{ and } J \subseteq (m]_L \cup (n]_L)$$

If L is a complete algebraic and dually algebraic lattice, then, by the above, the smallest weak dicomplementation on L is of the form $({}^{\Delta J}, {}^{\nabla M})$ for the join–dense subset $J = \mathrm{Sji}(L)$ and the meet–dense subset $M = \mathrm{Smi}(L)$ of L, therefore:

- L has nontrivial weak complementations iff $({}^{\Delta \mathrm{Sji}(L)}, {}^{\nabla \mathrm{Smi}(L)})$ is not the single weak complementation on L iff condition $\neg sg\Delta(L, \mathrm{Sji}(L), \mathrm{Smi}(L))$ is satisfied.

In particular, if L is a finite lattice, then:

- L has nontrivial weak complementations iff condition $\neg sg\Delta(L, \mathrm{Ji}(L), \mathrm{Mi}(L) \setminus \{1\})$ is satisfied.

And, of course, dually for the dual weak complementations. All the following results on weak complementations involving condition $\neg sg\Delta(L, J, M)$ can be dualized, using condition $\neg sg\Delta(L^d, M, J)$ in results on dual weak complementations.

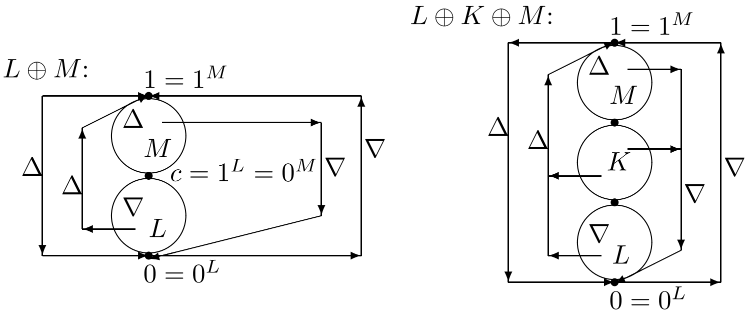
Remark 3.1 It is routine to prove that a lattice congruence of a bounded lattice L preserves the trivial weak complementation ${}^{\Delta L}$ on L iff its 1–class is a singleton and, dually, it preserves the trivial dual weak complementation ${}^{\nabla L}$ on L iff its 0–class is a singleton, therefore, with the notation for congruence lattices in Section 2:

- $\mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L}) = \mathrm{Con}_1(L) \cup \{L^2\}$ and $\mathrm{Con}_{\mathbb{DWCL}}(L, {}^{\nabla L}) = \mathrm{Con}_0(L) \cup \{L^2\}$, so $\mathrm{Con}_{\mathbb{WDL}}(L, {}^{\Delta L}, {}^{\nabla L}) = \mathrm{Con}_{01}(L) \cup \{L^2\}$;

thus $\mathrm{Con}_{\mathbb{WCL}1}(L) = \mathrm{Con}_1(L)$, $\mathrm{Con}_{\mathbb{DWCL}0}(L) = \mathrm{Con}_0(L)$ and $\mathrm{Con}_{\mathbb{WDL}01}(L) = \mathrm{Con}_{\mathbb{WDL}0}(L) = \mathrm{Con}_{\mathbb{WDL}1}(L) = \mathrm{Con}_{01}(L)$.

4 Weak Dicomplementations on Glued Sums

Let L and M be nonsingleton bounded lattices and let us consider the glued sum $L \oplus M$, with $L \cap M = \{c\}$.



Let $({}^\Delta, {}^\nabla)$ be a weak dicomplementation on $L \oplus M$. Since the lattice M is nonsingleton and $L \oplus M$ satisfies $x \vee x^\Delta \approx 1$, it follows that $a^\Delta = 1$ for all $a \in L$. We have $1^\Delta = 0$ and, clearly, ${}^\Delta|_{M\setminus\{1\}}$ is the restriction to $M \setminus \{1\}$ of a weak complementation on M, that we will denote by ${}^{\Delta_2}$, so that ${}^\Delta$ is trivial iff ${}^{\Delta_2}$ is trivial, and $L \oplus M$ has nontrivial weak complementations iff M does. Dually, ${}^\nabla$ must be defined by: $b^\nabla = 0$ for all $b \in M$, $0^\nabla = 1$ and ${}^\nabla|_{L\setminus\{0\}} = {}^{\nabla_1}|_{L\setminus\{0\}}$ for some dual weak complementation ${}^{\nabla_1}$ on L, so that ${}^\nabla$ is trivial iff ${}^{\nabla_1}$ is trivial, and $L \oplus M$ has nontrivial dual weak complementations iff L does. Consequently, $L \oplus M$ has nontrivial weak dicomplementations iff M has nontrivial weak complementations or L has nontrivial dual weak complementations. Thus:

Remark 4.1 For any bounded lattice A, $A \oplus \mathcal{C}_2$ has no nontrivial weak complementations, $\mathcal{C}_2 \oplus A$ has no nontrivial dual weak complementations and $\mathcal{C}_2 \oplus A \oplus \mathcal{C}_2$ has no nontrivial weak dicomplementations.

Lemma 4.1 *For any nonsingleton bounded lattices L and M*: $\mathrm{Con}_{\mathbb{WCL}}(L \oplus M) = \{\alpha \oplus \beta \mid \alpha \in \mathrm{Con}(L), \beta \in \mathrm{Con}_{\mathbb{WCL}1}(M)\} \cup \{(L \oplus M)^2\} \cong (\mathrm{Con}(L) \times \mathrm{Con}_{\mathbb{WCL}1}(M)) \oplus \mathcal{C}_2$.

Dually, $\mathrm{Con}_{\mathbb{DWCL}}(L \oplus M) = \{\alpha \oplus \beta \mid \alpha \in \mathrm{Con}_{\mathbb{DWCL}0}(L), \beta \in \mathrm{Con}(M)\} \cup \{(L \oplus M)^2\} \cong (\mathrm{Con}_{\mathbb{DWCL}0}(L) \times \mathrm{Con}(M)) \oplus \mathcal{C}_2$, *so* $\mathrm{Con}_{\mathbb{WDL}}(L \oplus M) = \{\alpha \oplus \beta \mid \alpha \in \mathrm{Con}_{\mathbb{WCL}0}(L), \beta \in \mathrm{Con}_{\mathbb{DWCL}1}(M)\} \cup \{(L \oplus M)^2\} \cong (\mathrm{Con}_{\mathbb{WCL}0}(L) \times \mathrm{Con}_{\mathbb{DWCL}1}(M)) \oplus \mathcal{C}_2$.

Proof We use the notations above for the (dual) weak complementations.

$\mathrm{Con}(L \oplus M) = \{\alpha \oplus \beta \mid \alpha \in \mathrm{Con}(L), \beta \in \mathrm{Con}(M)\}$. Let $\alpha \in \mathrm{Con}(L)$ and $\beta \in \mathrm{Con}(M)$. Of course, $\alpha \oplus \beta \in \mathrm{Con}_{\mathbb{WCL}}(L \oplus M)$ iff, for all $x, y \in L \oplus M$, whenever $x(\alpha \oplus \beta)y$, it follows that $x^\Delta(\alpha \oplus \beta)y^\Delta$.

If $x, y \in L$, then $x^{\Delta} = 1 = y^{\Delta}$, so the preservation of $^{\Delta}$ is trivially satisfied by $\alpha \oplus \beta$ in this case.

If $x, y \in M \setminus \{1\}$, then $x^{\Delta}(\alpha \oplus \beta)y^{\Delta}$ iff $x^{\Delta_2}\beta y^{\Delta_2}$, therefore, if the 1–class of β is a singleton, then $\alpha \oplus \beta$ preserves the $^{\Delta}$ iff β preserves the $^{\Delta_2}$.

If $x = 1$ and $y \in M \setminus \{1\}$ are such that $x(\alpha \oplus \beta)y$, case in which the 1–class of β is not a singleton, then $x^{\Delta}(\alpha \oplus \beta)y^{\Delta}$ iff $0 = 1^{\Delta}(\alpha \oplus \beta)y^{\Delta_2} \in M$, which is equivalent to $\alpha = L^2$ and $c\beta y^{\Delta_2}$, the latter of which holds when β preserves the $^{\Delta_2}$.

If $x = 1$ and $y \in M$, then $x = 1(\alpha \oplus \beta)y$ iff $\beta = M^2$ and $c\alpha y$, and, if $\alpha \oplus \beta$ preserves the $^{\Delta}$, then $0 = 1^{\Delta}(\alpha \oplus \beta)y^{\Delta} = 1$, hence $\alpha \oplus \beta = (L \oplus M)^2$, thus we also have $\alpha = L^2$.

If $x \in M \setminus \{1\}$ and $y \in M$ are such that $x(\alpha \oplus \beta)y$, so that $x\beta c$, then $x^{\Delta_2} = x^{\Delta}(\alpha \oplus \beta)y^{\Delta} = 1$ holds when β preserves the $^{\Delta_2}$.

Since the glued sums of congruences of the forms above clearly preserve the $^{\Delta}$, we have the equivalence: $\alpha \oplus \beta$ preserves the $^{\Delta}$ iff β preserves the $^{\Delta_2}$ and, whenever the 1–class of β is not a singleton, we have $\alpha = L^2$.

Remark 4.2 By Lemma 4.1 and the associativity of the glued sum, we obtain that, for any bounded lattice K and any nonsingleton bounded lattices L and M: $\mathrm{Con}_{\mathbb{WDL}}(L \oplus K \oplus M) = \{\alpha \oplus \theta \oplus \beta \mid \alpha \in \mathrm{Con}_{\mathbb{WCL}0}(L), \theta \in \mathrm{Con}(K), \quad \beta \in \mathrm{Con}_{\mathbb{DWCL}1}(M)\}$ $\cup\{(L \oplus K \oplus M)^2\} \cong (\mathrm{Con}_{\mathbb{WCL}0}(L) \times \mathrm{Con}(K) \times \mathrm{Con}_{\mathbb{DWCL}1}(M)) \oplus \mathcal{C}_2$.

5 Weak Dicomplementations on Atomic or Coatomic Lattices

Remark 5.1 Since $\mathbb{WCL} \vDash x \vee x^{\Delta} \approx 1$ and $\mathbb{DWCL} \vDash x \wedge x^{\nabla} \approx 0$, it follows that bounded lattices with the 1 join–irreducible can only be endowed with the trivial weak complementation, while bounded lattices with the 0 meet–irreducible can only be endowed with the trivial dual weak complementation, hence bounded lattices with the 0 meet–irreducible and the 1 join–irreducible can only be endowed with the trivial weak dicomplementation.

In particular, any bounded chain can only be endowed with the trivial weak dicomplementation.

Also, in the particular cases when 1 is strictly join–irreducible, respectively 0 is strictly meet–irreducible, we obtain Remark 4.1.

Since weak complementations and dual weak complementations are order–reversing: a weak complementation $^{\Delta}$ on a coatomic bounded lattice L is nontrivial iff $a^{\Delta} < 1$ for some $a \in \mathrm{CoAt}(L)$, which implies that $a^{\Delta} \leq b$ for some $b \in \mathrm{CoAt}(L) \setminus \{a\}$ since L is coatomic and $a \vee a^{\Delta} = 1$. And, of course, dually for dual weak complementations on atomic bounded lattices and the atoms of such lattices.

If L is a bounded lattice and $a, b \in L \setminus \{0, 1\}$ such that $a \neq b$, let us denote by $^{\Delta_{a,b}} : L \to L$ the operation defined by: $1^{\Delta_{a,b}} = 0$, $a^{\Delta_{a,b}} = b$, $b^{\Delta_{a,b}} = a$ and $x^{\Delta_{a,b}} = 1$ for all $x \in L \setminus \{1, a, b\}$. Since weak complementations are order–reversing, whenever $^{\Delta_{a,b}}$ is a weak complementation on L, a and b are coatoms of L, and, of course, since $a, b \in L \setminus \{0, 1\}$, we have $^{\Delta_{a,b}} \neq ^{\Delta L}$. With this notation, we have the following.

Proposition 5.1

1. *If L is a coatomic bounded lattice with exactly two coatoms a and b, then $^{\Delta_{a,b}} = ^{\Delta L \setminus \{1\}}$ is a representable nontrivial weak complementation on L.*

2. *If L is a bounded distributive lattice with at least two coatoms and a and b are distinct coatoms of L, then $^{\Delta a,b}$ is a nontrivial weak complementation on L. Furthermore, if* $\mathrm{Ji}(L)$ *is join–dense in L, in particular if L is finite, then $^{\Delta a,b} = ^{\Delta \mathrm{Ji}(L) \cup \{a,b\}}$.*

Dually for atoms and dual weak complementations.

Proof Let L be a bounded lattice.

(i) If L is coatomic and $\mathrm{CoAt}(L) = \{a, b\}$ with $a \neq b$, then clearly $(a]_L \cup (b]_L = L \setminus \{1\}$ is a join–dense subset of L and $^{\Delta a,b} = ^{\Delta L \setminus \{1\}}$.

(ii) Now assume that L is distributive and $\mathrm{CoAt}(L) \supseteq \{a, b\}$ with $a \neq b$. Then the operation $^{\Delta a,b}$ reverses the lattice order of L and satisfies $x^{\Delta a,b \Delta a,b} \leq x$ and $x \vee x^{\Delta a,b} = 1$ for all $x \in L$, therefore it is a weak complementation on L since L is distributive.

Any $x \in \mathrm{Ji}(L)$ satisfies $x \leq a$ iff $x \nleq b$ since $a \vee b = 1 \geq x$ (hence x is a $\vee$*–primary element* w.r.t. $^{\Delta a,b}$ [15, Section 3.1]), from which it is routine to prove that $^{\Delta a,b} = ^{\Delta \mathrm{Ji}(L) \cup \{a,b\}}$.

Remark 5.2 If L is a bounded lattice and $a, b \in L \setminus \{0, 1\}$ with $a \neq b$ are such that $^{\Delta a,b}$ is a weak complementation on L, then it is routine to prove that $\mathrm{Con}_{\mathbb{WCL}}(L, ^{\Delta a,b}) = \mathrm{Con}_{ab1}(L) \cup \{\theta \in \mathrm{Con}(L) \mid a/\theta = \{a, 1\}, \quad (0, b) \in \theta\} \cup \{\theta \in \mathrm{Con}(L) \mid b/\theta = \{b, 1\}, (0, a) \in \theta\} \cup \{L^2\}$.

Example 5.1 Clearly, there exist coatomic bounded lattices with nontrivial weak complementations having any number of coatoms greater than 2, as well as atomic bounded lattices with nontrivial dual weak complementations having any number of atoms greater than 2, Boolean algebras being the simplest example. We can also construct such lattices using direct products, since, for any nonsingleton weakly complemented lattices $(A, ^{\Delta_1})$ and $(B, ^{\Delta_2})$, the product weak complementation $^{\Delta_1 \times \Delta_2}$ on $A \times B$ is nontrivial, and, of course, $|\mathrm{CoAt}(A \times B)| = |\mathrm{CoAt}(A)| + |\mathrm{CoAt}(B)|$ and $\mathrm{Con}_{\mathbb{WCL}}(A \times B, ^{\Delta_1 \times \Delta_2}) \cong \mathrm{Con}_{\mathbb{WCL}}(A, ^{\Delta_1}) \times \mathrm{Con}_{\mathbb{WCL}}(B, ^{\Delta_2})$. Dually in $\mathbb{DWCL}$ and thus similarly in $\mathbb{WDL}$.

Note, also, that, for any bounded lattice L, the single coatom of $\mathrm{Con}_{\mathbb{WCL}}(L, ^{\Delta L})$, $\mathrm{Con}_{\mathbb{DWCL}}(L, ^{\nabla L})$, respectively $\mathrm{Con}_{\mathbb{WDL}}(L, ^{\Delta L}, ^{\nabla L})$ is $\max(\mathrm{Con}_1(L))$, $\max(\mathrm{Con}_0(L))$, respectively $\max(\mathrm{Con}_{01}(L))$, thus these lattices are directly irreducible. Consequently, if $\mathrm{Con}_{\mathbb{WCL}}(L)$, $\mathrm{Con}_{\mathbb{DWCL}}(L)$, respectively $\mathrm{Con}_{\mathbb{WDL}}(L)$ is directly reducible, in particular if L is directly reducible in $\mathbb{WCL}$, $\mathbb{DWCL}$, respectively $\mathbb{WDL}$, then the weak complementation, respectively the dual weak complementation, respectively the weak dicomplementation of L is nontrivial.

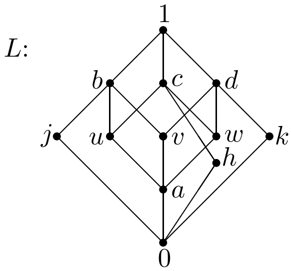
On the other hand, there exist bounded lattices with no nontrivial weak complementations having any number κ of coatoms other than 2, for instance $\mathcal{M}_\kappa$, as well as bounded lattices with no nontrivial dual weak complementations having any number λ of atoms other than 2, for instance $\mathcal{M}_\lambda$, and bounded lattices with no nontrivial weak dicomplementations having any (equal or distinct) numbers κ and λ of atoms and coatoms, respectively, other than 2, for instance $\mathcal{M}_\kappa \oplus \mathcal{M}_\lambda$; see Section 4.

Remark 5.3 For any coatomic bounded lattice L, any $J \subseteq L$ and any $M \subseteq L \setminus \{1\}$ such that $\mathrm{CoAt}(L) \subseteq M$, condition $\neg sg\Delta(L, J, M)$ is clearly equivalent to $\neg sg\Delta(L, J, \mathrm{CoAt}(L))$.

Example 5.2 Remark 5.3 provides us with an easy construction one can apply to coatomic complete algebraic and dually algebraic lattices L having at least three distinct coatoms in order to transform them into bounded lattices without nontrivial weak complementations having the same number of coatoms: for at least three distinct coatoms a, b, c of L, choose elements p, q, r of L such that $p < a, q < b, r < c$, and replace each of the intervals $[p,a]_L, [q,b]_L, [r,c]_L$ with its horizontal sum with a complete algebraic and dually algebraic lattice having at least one strictly join irreducible other than its top element. The resulting bounded lattice M will have the same coatoms as L, it will be complete, algebraic and dually algebraic, and it will clearly fail condition $\neg sg\Delta(M, \mathrm{Sji}(M), \mathrm{CoAt}(M) = \mathrm{CoAt}(L))$.

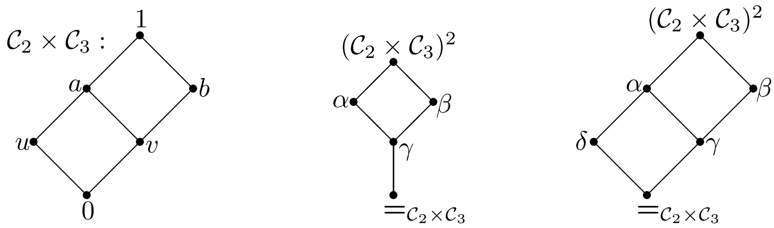
In particular, the construction above applied to a finite lattice with at least three coatoms, considering horizontal sums with finite lattices with join–irreducibles other than their lattice bounds, in particular with finite chains of lengths at least three, produces finite lattices with the same number of coatoms and without nontrivial weak complementations.

Here is the previous construction applied to $\mathcal{C}_2 \oplus \mathcal{C}_2^3$, which, by Section 4, has nontrivial weak complementations since $\mathcal{C}_2^3$ does, with the intervals given by the filters generated by each of its coatoms, turned into their horizontal sums with the three–element chain; the resulting lattice L has no nontrivial weak complementation, since it clearly fails condition $\neg sg\Delta(L, \mathrm{Ji}(L), \mathrm{CoAt}(L))$:



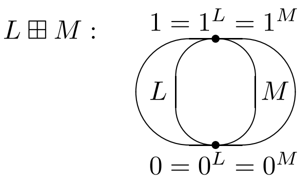
Example 5.3 By [7, 15], the only weak complementations on the direct product of chains $\mathcal{C}_2 \times \mathcal{C}_3$, with the elements denoted as in the leftmost Hasse diagram below, are:

- the trivial weak complementation ${}^{\Delta_{\mathcal{C}_2\times\mathcal{C}_3}}$, w.r.t. which $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2 \times \mathcal{C}_3, {}^{\Delta_{\mathcal{C}_2\times\mathcal{C}_3}}) = \mathrm{Con}_1(\mathcal{C}_2 \times \mathcal{C}_3) \cup \{(\mathcal{C}_2 \times \mathcal{C}_3)^2\} = \{=_{\mathcal{C}_2\times\mathcal{C}_3}, \gamma, (\mathcal{C}_2 \times \mathcal{C}_3)^2\} \cong \mathcal{C}_3$, where $\gamma = \mathrm{eq}(\{0,v\},\{u,a\}\{b\},\{1\})$, thus $\mathrm{Con}_{\mathbb{WCL}1}(\mathcal{C}_2 \times \mathcal{C}_3, {}^{\Delta_{\mathcal{C}_2\times\mathcal{C}_3}}) = \{=_{\mathcal{C}_2\times\mathcal{C}_3}, \gamma\} \cong \mathcal{C}_2$;
- with the notation above, ${}^{\Delta_{a,b}} = {}^{\Delta_{(\mathcal{C}_2\times\mathcal{C}_3)\setminus\{1\}}}$, w.r.t. which $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2 \times \mathcal{C}_3, {}^{\Delta_{a,b}}) = \{=_{\mathcal{C}_2\times\mathcal{C}_3}, \alpha, \beta, \gamma, (\mathcal{C}_2 \times \mathcal{C}_3)^2\} \cong \mathcal{C}_2 \oplus \mathcal{C}_2^2$, as in the middle diagram below, where $\alpha = \mathrm{eq}(\{0,u,v,a\},\{b,1\})$, $\beta = \mathrm{eq}(\{0,v,b\},\{u,a,1\})$ and $\gamma = \alpha \cap \beta$ is as above, thus $\mathrm{Con}_{\mathbb{WCL}1}(\mathcal{C}_2 \times \mathcal{C}_3, {}^{\Delta_{a,b}}) = \{=_{\mathcal{C}_2\times\mathcal{C}_3}, \gamma\} \cong \mathcal{C}_2$;
- the direct product ${}^{\Delta_{\mathcal{C}_2}\times\Delta_{\mathcal{C}_3}}$ of the trivial weak complementations ${}^{\Delta_{\mathcal{C}_2}}$ and ${}^{\Delta_{\mathcal{C}_3}}$ on the chains $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively, defined by $1^{\Delta_{\mathcal{C}_2}\times\Delta_{\mathcal{C}_3}} = 0$, $a^{\Delta_{\mathcal{C}_2}\times\Delta_{\mathcal{C}_3}} = u^{\Delta_{\mathcal{C}_2}\times\Delta_{\mathcal{C}_3}} = b$, $b^{\Delta_{\mathcal{C}_2}\times\Delta_{\mathcal{C}_3}} = u$ and $v^{\Delta_{\mathcal{C}_2}\times\Delta_{\mathcal{C}_3}} = 0^{\Delta_{\mathcal{C}_2}\times\Delta_{\mathcal{C}_3}} = 1$, w.r.t. which $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2 \times \mathcal{C}_3, {}^{\Delta_{\mathcal{C}_2}\times\Delta_{\mathcal{C}_3}}) \cong \mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2, {}^{\Delta_{\mathcal{C}_2}}) \times \mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_3, {}^{\Delta_{\mathcal{C}_3}}) = (\mathrm{Con}_1(\mathcal{C}_2) \cup \{\mathcal{C}_2^2\}) \times (\mathrm{Con}_1(\mathcal{C}_3) \cup \{\mathcal{C}_3^2\}) \cong (\mathcal{C}_1 \oplus \mathcal{C}_2) \times (\mathcal{C}_2 \oplus \mathcal{C}_2) \cong \mathcal{C}_2 \times \mathcal{C}_3$, specifically $\mathrm{Con}(\mathcal{C}_2 \times \mathcal{C}_3) = \{=_{\mathcal{C}_2\times\mathcal{C}_3}, \alpha, \beta, \gamma, \delta, (\mathcal{C}_2 \times \mathcal{C}_3)^2\}$, as in the rightmost diagram below, where $\delta = \mathrm{eq}(\{0,u\},\{v,a\},\{b,1\})$; so $\mathrm{Con}_{\mathbb{WCL}1}(\mathcal{C}_2 \times \mathcal{C}_3, {}^{\Delta_{\mathcal{C}_2}\times\Delta_{\mathcal{C}_3}}) \cong \mathrm{Con}_1(\mathcal{C}_2) \times \mathrm{Con}_1(\mathcal{C}_3) \cong \mathcal{C}_1 \times \mathcal{C}_2 \cong \mathcal{C}_2$; notice that ${}^{\Delta_{\mathcal{C}_2}\times\Delta_{\mathcal{C}_3}} = {}^{\Delta_{\{u,v,b\}}} = {}^{\Delta_{\mathrm{Ji}(\mathcal{C}_2\times\mathcal{C}_3)}} = {}^{\Delta_{\mathrm{Sji}(\mathcal{C}_2\times\mathcal{C}_3)}}$.



6 Weak Dicomplementations on Horizontal Sums

Throughout this section, L and M will be bounded lattices with $|L| > 2$ and $|M| > 2$ and we will consider their horizontal sum $L \boxplus M$. Unless mentioned otherwise, $(^{\Delta_1}, ^{\nabla_1})$, $(^{\Delta_2}, ^{\nabla_2})$ and $(^{\Delta}, ^{\nabla})$ will be arbitrary weak dicomplementations on L, M and $L \boxplus M$, respectively.



Of course, L and M are bounded sublattices of $L \boxplus M$, and $L \boxplus M$ is complete, atomic, respectively coatomic iff L and M are complete, atomic, respectively coatomic. Also, $\mathrm{Ji}(L \boxplus M) = (\mathrm{Ji}(L) \cup \mathrm{Ji}(M)) \setminus \{1\}$, $\mathrm{Sji}(L \boxplus M) = (\mathrm{Sji}(L) \cup \mathrm{Sji}(M)) \setminus \{1\}$, $\mathrm{Mi}(L \boxplus M) = (\mathrm{Mi}(L) \cup \mathrm{Mi}(M)) \setminus \{0\}$ and $\mathrm{Smi}(L \boxplus M) = (\mathrm{Smi}(L) \cup \mathrm{Smi}(M)) \setminus \{0\}$, so $L \boxplus M$ is algebraic, respectively dually algebraic iff L and M are algebraic, respectively dually algebraic.

Lemma 6.1 *For all $x \in L \setminus \{0, 1\}$ and all $y \in M \setminus \{0, 1\}$, we have $x^{\nabla} \le y \le x^{\Delta}$ and $y^{\nabla} \le x \le y^{\Delta}$, in particular $x^{\Delta}, x^{\nabla} \in M$ and $y^{\Delta}, y^{\nabla} \in L$.*

Proof We have: $0 \ne x = (x \wedge y) \vee (x \wedge y^{\Delta}) = 0 \vee (x \wedge y^{\Delta}) = x \wedge y^{\Delta}$, hence $x \le y^{\Delta}$. Analogously for x^{Δ}, and dually for the dual weak complementation.

Recall from [18] that:

- $\mathrm{Con}_{01}(L \boxplus M) = \{\alpha \boxplus \beta \mid \alpha \in \mathrm{Con}_{01}(L), \beta \in \mathrm{Con}_{01}(M)\} \cong \mathrm{Con}_{01}(L) \times \mathrm{Con}_{01}(M)$;
- $\mathrm{Con}_0(L \boxplus M) = \mathrm{Con}_1(L \boxplus M) = \mathrm{Con}_{01}(L \boxplus M) \subset \mathrm{Con}(L \boxplus M) \subseteq \mathrm{Con}_{01}(L \boxplus M) \cup \{\mathrm{eq}(L \setminus \{0\}, M \setminus \{1\}), \mathrm{eq}(L \setminus \{1\}, M \setminus \{0\}), (L \boxplus M)^2\}$, and: $\mathrm{eq}(L \setminus \{0\}, M \setminus \{1\}) \in \mathrm{Con}(L \boxplus M)$ iff $0 \in \mathrm{Mi}(L)$ and $1 \in \mathrm{Ji}(M)$, while $\mathrm{eq}(L \setminus \{1\}, M \setminus \{0\}) \in \mathrm{Con}(L \boxplus M)$ iff $1 \in \mathrm{Ji}(L)$ and $0 \in \mathrm{Mi}(M)$.

Remark 6.1 $L \in \mathrm{S}_{\mathbb{WCL}}(L \boxplus M)$ iff $^{\Delta_1}$ is the trivial weak complementation $^{\Delta L}$ and $^{\Delta}|_L = ^{\Delta_1}$.

Indeed, by Lemma 6.1, if $L \in \mathrm{S}_{\mathbb{WCL}}(L \boxplus M)$, then, for any $x \in L$, we have $x^{\Delta_1} = x^{\Delta} \in M \cap L = \{0, 1\}$, hence $^{\Delta_1}$ is trivial and $^{\Delta}|_L = ^{\Delta_1}$. The converse is clear. Of course, similarly for M.

Hence: $(L \boxplus M,^{\Delta})$ is the horizontal sum of the algebras $(L,^{\Delta_1})$ and $(M,^{\Delta_2})$ from $\mathbb{WCL}$, that is $L, M \in \mathrm{S}_{\mathbb{WCL}}(L \boxplus M)$, iff $^{\Delta}$, $^{\Delta_1}$ and $^{\Delta_2}$ are trivial.

Dually for $\mathbb{DWCL}$, thus similarly in $\mathbb{WDL}$.

By the above, $\mathrm{Con}_{\mathbb{WDL}}(L \boxplus M,^{\Delta L\boxplus M},^{\nabla L\boxplus M}) = \mathrm{Con}_{\mathbb{WCL}}(L \boxplus M,^{\Delta L\boxplus M}) = \mathrm{Con}_{\mathbb{DWCL}}(L \boxplus M,^{\nabla L\boxplus M}) = \mathrm{Con}_{01}(L \boxplus M) \cup \{(L \boxplus M)^2\} = \{\alpha \boxplus \beta \mid \alpha \in \mathrm{Con}_{01}(L) = \mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta L},^{\nabla L}) \setminus \{L^2\} = \mathrm{Con}_{\mathbb{WCL}0}(L,^{\Delta L}) = \mathrm{Con}_{\mathbb{DWCL}1}(L,^{\nabla L}), \beta \in \mathrm{Con}_{01}(M) = \mathrm{Con}_{\mathbb{WDL}}(M,^{\Delta M},^{\nabla M}) \setminus \{M^2\} = \mathrm{Con}_{\mathbb{WCL}0}(M,^{\Delta M}) = \mathrm{Con}_{\mathbb{DWCL}1}(M,^{\nabla M})\} \cup \{(L \boxplus M)^2\} \cong (\mathrm{Con}_{01}(L) \times \mathrm{Con}_{01}(M)) \oplus \mathcal{C}_2$, so the horizontal sum cancels congruences in $\mathbb{WCL}$ and $\mathbb{DWCL}$, while keeping congruences in $\mathbb{WDL}$ in place.

For example, since $\mathcal{C}_2^2 = \mathcal{C}_3 \boxplus \mathcal{C}_3$, we have: $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2^2,^{\Delta \mathcal{C}_2^2}) = \mathrm{Con}_{\mathbb{DWCL}}(\mathcal{C}_2^2,^{\nabla \mathcal{C}_2^2}) = \mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_2^2,^{\Delta \mathcal{C}_2^2},^{\nabla \mathcal{C}_2^2}) = \mathrm{Con}_{01}(\mathcal{C}_2^2) \cup \{(\mathcal{C}_2^2)^2\} = \{=_{\mathcal{C}_2^2}, (\mathcal{C}_2^2)^2\}$.

Lemma 6.2 *If* 1 *is not strictly join–irreducible in L, then* $^{\Delta}\,|_M$ *is the trivial weak complementation* $^{\Delta M}$ *on M. Dually for* 0 *and the dual weak complementation.*

Proof Assume that $1 \notin \mathrm{Sji}(L)$, so that $1 \notin \mathrm{Ji}(L)$ or $1 \in \mathrm{Ji}(L) \setminus \mathrm{Sji}(L)$; of course, in the latter case, L has to be infinite. Let $y \in M \setminus \{1\}$.

By Lemma 6.1, if 1 is join–reducible in L, so that $1 = a \vee b$ for some $a, b \in L \setminus \{1\}$, then $y^{\Delta} \geq a$ and $y^{\Delta} \geq b$, thus $y^{\Delta} = 1$, so that $^{\Delta}\,|_M = ^{\Delta M}$.

If 1 is join–irreducible, but not strictly join–irreducible in L, then 1 has no lower covers in L. Then, if $y^{\Delta} < 1$, we would have $y^{\Delta} \nprec 1$, so that there would exist some $z \in L$ such that $y^{\Delta} < z < 1$, thus $y^{\Delta} \ngeq z \in L \setminus \{1\}$, contradicting Lemma 6.1. Hence $^{\Delta} = ^{\Delta M}$.

Proposition 6.1 $L \boxplus M$ *has nontrivial weak complementations iff* 1 *is strictly join–irreducible in each of the lattices L and M, case in which* $L \boxplus M$ *has only these two weak complementations, both of which are representable:* $^{\Delta L\boxplus M}$ *and* $^{\Delta(L\boxplus M)\setminus\{1\}}$*, the latter of which is nontrivial and, in the particular case when L and M are complete and dually algebraic, coincides to* $^{\Delta \mathrm{Sji}(L\boxplus M)}$*. Dually for dual weak complementations.*

Proof If $1 \in \mathrm{Sji}(L) \cap \mathrm{Sji}(M)$, then $L \boxplus M$ is a coatomic lattice with exactly two coatoms, $1^{-L} \in L$ and $1^{-M} \in M$, thus, by Proposition 5.1, (i), with the notation in Section 5, it has the nontrivial weak complementation $^{\Delta 1^{-L},1^{-M}} = ^{\Delta(L\boxplus M)\setminus\{1\}}$.

Now assume that $L \boxplus M$ has a nontrivial weak complementation $^{\Delta}$ and let $x \in (L \boxplus M) \setminus \{1\}$ such that $x^{\Delta} \neq 1$. Then $x \neq 0$ and w.l.g. we may assume that $x \in L$. Then, by Lemma 6.2, $1 \in \mathrm{Sji}(M)$ and $x^{\Delta} = 1^{-M}$, the unique coatom of M. Hence, for all $y \in L \setminus \{1\}$, we have $y = (y \wedge x) \vee (y \wedge x^{\Delta}) = (y \wedge x) \vee (y \wedge 1^{-M}) = (y \wedge x) \vee 0 = y \wedge x$, thus $y \leq x$. Therefore $x = \max(L \setminus \{1\})$, so $1 \in \mathrm{Sji}(L)$ and $x = 1^{-L}$, the unique coatom of L. If $(1^{-L})^{\Delta} = 1^{-M}$ and $(1^{-M})^{\Delta} = 1$, then $(1^{-L})^{\Delta\Delta} = 1 \nleq 1^{-L}$, which contradicts the definition of a weak complementation. We get a similar contradiction if we assume that $(1^{-L})^{\Delta} = 1$ and $(1^{-M})^{\Delta} = 1^{-L}$. Consequently, $^{\Delta} = ^{\Delta(L\boxplus M)\setminus\{1\}}$.

Either by the property of weak complementations on complete dually algebraic lattices at the end of Section 3 or directly from the definition of such a lattice, we get the last statement in the enunciation.

Corollary 6.1 *If K is a bounded lattice with* $|K| > 2$*, then* $K \boxplus L \boxplus M$ *can only be endowed with the trivial weak dicomplementation.*

Proposition 6.2 *If $L \boxplus M$ has a nontrivial weak complementation, then, if we denote by $\phi = \mathrm{eq}(L \setminus \{0\}, M \setminus \{1\})$ and $\psi = \mathrm{eq}(L \setminus \{1\}, M \setminus \{0\})$, we have:*

1. *if $L \boxplus M = \{0, 1^{-L}, 1^{-M}, 1\} \cong \mathcal{C}_2^2$, or, equivalently, $L = \{0, 1^{-L}, 1\} \cong \mathcal{C}_3$ and $M = \{0, 1^{-M}, 1\} \cong \mathcal{C}_3$, then $^{\Delta(L \boxplus M) \setminus \{1\}}$ is the Boolean complementation and:* $\mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, ^{\Delta(L \boxplus M) \setminus \{1\}}) = \{=_{L \boxplus M}, \phi, \psi, (L \boxplus M)^2\} \cong \mathcal{C}_2^2$;
2. *if $L \cong \mathcal{C}_3$ and $|M| > 3$, then:* $\mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, ^{\Delta(L \boxplus M) \setminus \{1\}}) = \{=_L \boxplus (\delta \oplus =_{\mathcal{C}_2}) \mid \delta \in \mathrm{Con}_{01}((1^{-M}]_M)\} \cup \{\phi, (L \boxplus M)^2\} \cong \mathrm{Con}_{01}((1^{-M}]_M) \oplus \mathcal{C}_3$;
3. *if $|L| > 3$ and $|M| > 3$, then:* $\mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, ^{\Delta(L \boxplus M) \setminus \{1\}}) = \{(\gamma \oplus =_{\mathcal{C}_2}) \boxplus (\delta \oplus =_{\mathcal{C}_2}) \mid \gamma \in \mathrm{Con}_{01}((1^{-L}]_L), \delta \in \mathrm{Con}_{01}((1^{-M}]_M)\} \cup \{(L \boxplus M)^2\} \cong (\mathrm{Con}_{01}((1^{-L}]_L) \times \mathrm{Con}_{01}((1^{-M}]_M)) \oplus \mathcal{C}_2$.
   Dually in $\mathbb{DWCL}$.

Proof Remark 6.1 gives us the congruences of the weakly complemented lattice $(L \boxplus M, ^{\Delta L \boxplus M})$.

Since $L \boxplus M$ admits other weak complementations except $^{\Delta L \boxplus M}$, by Proposition 6.1 we have $1 \in \mathrm{Sji}(L) \cap \mathrm{Sji}(M)$ and thus:

- $\phi \in \mathrm{Con}(L \boxplus M)$ iff $0 \in \mathrm{Ji}(L)$, while $\psi \in \mathrm{Con}(L \boxplus M)$ iff $0 \in \mathrm{Ji}(M)$;
- $L = (1^{-L}]_L \oplus \mathcal{C}_2$, so $\mathrm{Con}(L) = \{\gamma \oplus \zeta \mid \gamma \in \mathrm{Con}((1^{-L}]_L), \zeta \in \mathrm{Con}(\mathcal{C}_2)\} = \{\gamma \oplus =_{\mathcal{C}_2}, \gamma \oplus \mathcal{C}_2^2 \mid \gamma \in \mathrm{Con}((1^{-L}]_L)\}$;
- $M = (1^{-M}]_M \oplus \mathcal{C}_2$, so, analogously, $\mathrm{Con}(M) = \{\delta \oplus =_{\mathcal{C}_2}, \delta \oplus \mathcal{C}_2^2 \mid \delta \in \mathrm{Con}((1^{-M}]_M)\}$.

Let $\theta \in \mathrm{Con}(L \boxplus M) \setminus \{(L \boxplus M)^2\}$, arbitrary, and let us consider the following conditions on the lattice congruence θ:

$$(LM)(\theta) \quad (\forall x \in (L \boxplus M) \setminus \{1^{-L}, 1^{-M}\})(x \theta 1^{-L} \Rightarrow x^{\Delta(L \boxplus M) \setminus \{1\}} \theta 1^{-M})$$
$$(ML)(\theta) \quad (\forall x \in (L \boxplus M) \setminus \{1^{-L}, 1^{-M}\})(x \theta 1^{-M} \Rightarrow x^{\Delta(L \boxplus M) \setminus \{1\}} \theta 1^{-L})$$

Since $\theta \neq (L \boxplus M)^2$, we have $\theta = (\theta \cap L^2) \boxplus (\theta \cap M^2)$, with $\theta \cap L^2 \in \mathrm{Con}(L) \setminus \{L^2\}$ and $\theta \cap M^2 \in \mathrm{Con}(M) \setminus \{M^2\}$ since L and M are sublattices of $L \boxplus M$ and $(0, 1) \notin \theta$, so that $(0, 1) \notin \theta \cap L^2$ and $(0, 1) \notin \theta \cap M^2$. Thus, for any $x \in L \setminus \{0, 1\} = L \setminus M$ and any $y \in M \setminus \{0, 1\} = M \setminus L$, $(x, y) \notin \alpha \boxplus \beta$, so that $x/\theta = x/(\theta \cap L^2)$ and $y/\theta = y/(\theta \cap M^2)$, in particular, $(1^{-L}, 1^{-M}) \notin \theta$ and hence:

$$\theta \in \mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, ^{\Delta(L \boxplus M) \setminus \{1\}}) \text{ iff it satisfies conditions } (LM)(\theta) \text{ and } (ML)(\theta).$$

Now let $\alpha \in \mathrm{Con}_{01}(L)$ and $\beta \in \mathrm{Con}_{01}(M)$, so that $\alpha \boxplus \beta \in \mathrm{Con}_{01}(L \boxplus M) \subseteq \mathrm{Con}(L \boxplus M) \setminus \{(L \boxplus M)^2\}$, $(\alpha \boxplus \beta) \cap L^2 = \alpha$ and $(\alpha \boxplus \beta) \cap M^2 = \beta$, $x/\alpha \subseteq L \setminus \{0, 1\}$ for any $x \in L \setminus \{0, 1\}$ and $y/\beta \subseteq M \setminus \{0, 1\}$ for any $y \in M \setminus \{0, 1\}$. In particular, $1^{-L}/(\alpha \boxplus \beta) = 1^{-L}/\alpha \subseteq L \setminus \{0, 1\}$ and $1^{-M}/(\alpha \boxplus \beta) = 1^{-M}/\beta \subseteq M \setminus \{0, 1\}$.

If there exists an $x \in 1^{-L}/(\alpha \boxplus \beta) \setminus \{1^{-L}, 1^{-M}\} = \alpha$, then $x^{\Delta(L \boxplus M) \setminus \{1\}}/(\alpha \boxplus \beta) = 1/(\alpha \boxplus \beta) = \{1\} \neq 1^{-M}/\beta = 1^{-M}/(\alpha \boxplus \beta)$, so condition $(LM)(\alpha \boxplus \beta)$ fails. Clearly, if $1^{-L}/\alpha = \{1^{-L}\}$, then condition $(LM)(\alpha \boxplus \beta)$ is satisfied.

Similarly, condition $(ML)(\alpha \boxplus \beta)$ is satisfied iff $1^{-M}/\beta = \{1^{-M}\}$.

Therefore: $\alpha \boxplus \beta \in \mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, ^{\Delta(L \boxplus M) \setminus \{1\}})$ iff both conditions $(LM)(\alpha \boxplus \beta)$ and $(ML)(\alpha \boxplus \beta)$ are satisfied iff $1^{-L}/\alpha = \{1^{-L}\}$ and $1^{-M}/\beta = \{1^{-M}\}$ iff $\alpha = \gamma \oplus =_{\mathcal{C}_2}$ for some $\gamma \in \mathrm{Con}_{01}((1^{-L}]_L)$ and $\beta = \delta \oplus =_{\mathcal{C}_2}$ for some $\delta \in \mathrm{Con}_{01}((1^{-M}]_M)$, by the above, hence:

$$\mathrm{Con}_{\mathbb{WCL}01}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}}) = \{(\gamma \oplus =_{\mathcal{C}_2}) \boxplus (\delta \oplus =_{\mathcal{C}_2}) \mid \gamma \in \mathrm{Con}_{01}((1^{-L}]_L), \delta \in \mathrm{Con}_{01}((1^{-M}]_M)\}$$

$$\cong \mathrm{Con}_{01}((1^{-L}]_L) \times \mathrm{Con}_{01}((1^{-M}]_M).$$

Whenever $|L| > 3$, so that there exists an element $x \in L \setminus \{0, 1^{-L}, 1\} \subset 1^{-L}/\phi = L \setminus \{0\}$, we have $x^{\Delta(L\boxplus M)\setminus\{1\}} = 1 \notin 1^{-M}/\phi$, hence condition $(LM)(\phi)$ fails. If $L \cong \mathcal{C}_3$, so that $L = \{0, 1^{-L}, 1\}$ and thus $1^{-L}/\phi = \{1^{-L}, 1\}$, then $1^{\Delta(L\boxplus M)\setminus\{1\}} = 0 \in 1^{-M}/\phi$, so condition $(LM)(\phi)$ is satisfied. Similarly, condition $(ML)(\psi)$ is satisfied iff $M \cong \mathcal{C}_3$.

For every $x \in 1^{-M}/\phi \setminus \{1^{-M}\} = M \setminus \{1^{-M}, 1\}$, we have $x^{\Delta(L\boxplus M)\setminus\{1\}} = 1 \in L \setminus \{0\} = 1^{-L}/\phi$, thus condition $(ML)(\phi)$ is satisfied. Similarly, condition $(LM)(\psi)$ is satisfied.

Therefore: $\phi \in \mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}})$ iff conditions $(LM)(\phi)$ and $(ML)(\phi)$ are satisfied iff condition $(LM)(\phi)$ is satisfied iff $L \cong \mathcal{C}_3$.

Similarly: $\psi \in \mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}})$ iff conditions $(LM)(\psi)$ and $(ML)(\psi)$ are satisfied iff condition $(ML)(\psi)$ is satisfied iff $M \cong \mathcal{C}_3$.

Therefore we have the following cases. Note that, when $L \cong \mathcal{C}_3$, so that $(1^{-L}]_L \cong \mathcal{C}_2$, we have: $\mathrm{Con}_{01}(L) = \{=_L\} \cong \mathcal{C}_1 \cong \{=_{(1^{-L}]_L}\} = \mathrm{Con}_{01}((1^{-L}]_L) = \mathrm{Con}_0((1^{-L}]_L)$, and similarly for M.

(i) If $L \boxplus M = \{0, 1^{-L}, 1^{-M}, 1\} \cong \mathcal{C}_2^2$, then $\phi, \psi \in \mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}})$ and $\mathrm{Con}_{01}(L \boxplus M) = \{=_{L\boxplus M}\} \cong \mathcal{C}_1$.

(ii) If $L = \{0, 1^{-L}, 1\} \cong \mathcal{C}_3$, but $|M| > 3$, then $\phi \in \mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}})$, but $\psi \notin \mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}})$.

(iii) If $|L| > 3$ and $|M| > 3$, then $\phi, \psi \notin \mathrm{Con}_{\mathbb{WCL}}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}})$.

Hence the forms of the congruence lattices in the enunciation.

Corollary 6.2 *$L \boxplus M$ has nontrivial weak dicomplementations iff at least one of the following conditions holds*:

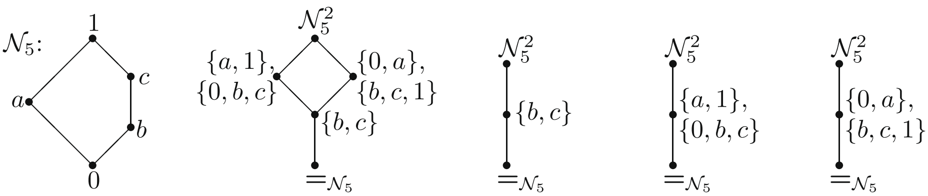
- 1 is strictly join–irreducible in both L and M;
- 0 is strictly meet–irreducible in both L and M.

The weak dicomplementations on $L \boxplus M$ are $({}^{\Delta}, {}^{\nabla})$, with ${}^{\Delta} \in \{{}^{\Delta L\boxplus M}, {}^{\Delta(L\boxplus M)\setminus\{1\}}\}$ if $1 \in \mathrm{Sji}(L) \cap \mathrm{Sji}(M)$ and ${}^{\Delta} = {}^{\Delta L\boxplus M}$ otherwise, and ${}^{\nabla} \in \{{}^{\nabla L\boxplus M}, {}^{\nabla(L\boxplus M)\setminus\{0\}}\}$ if $0 \in \mathrm{Smi}(L) \cap \mathrm{Smi}(M)$ and ${}^{\nabla} = {}^{\nabla L\boxplus M}$ otherwise, all of which are representable.

- If $L \boxplus M \cong \mathcal{C}_2^2$, which is equivalent to $L \cong M \cong \mathcal{C}_3$ and also to $1^{-L} = 0^{+L}$ and $1^{-M} = 0^{+M}$, then both ${}^{\Delta(L\boxplus M)\setminus\{1\}}$ and ${}^{\nabla(L\boxplus M)\setminus\{0\}}$ equal the Boolean complementation, so we have: $\mathrm{Con}_{\mathbb{WDL}}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}}, {}^{\nabla(L\boxplus M)\setminus\{0\}}) \cong \mathcal{C}_2^2$, $\mathrm{Con}_{\mathbb{WDL}}(L \boxplus M, {}^{\Delta L\boxplus M}, {}^{\nabla L\boxplus M}) = \mathrm{Con}_{\mathbb{WDL}}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}}, {}^{\nabla L\boxplus M}) = \mathrm{Con}_{\mathbb{WDL}}(L \boxplus M, {}^{\Delta L\boxplus M}, {}^{\nabla(L\boxplus M)\setminus\{0\}}) = \{=_{L\boxplus M}, (L \boxplus M)^2\} \cong \mathcal{C}_2$.
- If $L \boxplus M \ncong \mathcal{C}_2^2$ and $1 \in \mathrm{Sji}(L) \cap \mathrm{Sji}(M)$, then: $\mathrm{Con}_{\mathbb{WDL}}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}}, {}^{\nabla L\boxplus M}) = \{(\gamma \oplus =_{\mathcal{C}_2}) \boxplus (\delta \oplus =_{\mathcal{C}_2}) \mid \gamma \in \mathrm{Con}_{01}((1^{-L}]_L), \delta \in \mathrm{Con}_{01}((1^{-M}]_M)\} \cup \{(L \boxplus M)^2\}$ $\cong (\mathrm{Con}_{01}((1^{-L}]_L) \times \mathrm{Con}_{01}((1^{-M}]_M)) \oplus \mathcal{C}_2$.
- If $L \boxplus M \ncong \mathcal{C}_2^2$ and $0 \in \mathrm{Smi}(L) \cap \mathrm{Smi}(M)$, then: $\mathrm{Con}_{\mathbb{WDL}}(L \boxplus M, {}^{\Delta L\boxplus M}, {}^{\nabla(L\boxplus M)\setminus\{0\}}) = \{(=_{\mathcal{C}_2} \oplus \gamma) \boxplus (=_{\mathcal{C}_2} \oplus \delta) \mid \gamma \in \mathrm{Con}_{01}([0^{+L})_L), \delta \in \mathrm{Con}_{01}([0^{+M})_M)\} \cup \{(L \boxplus M)^2\}$ $\cong (\mathrm{Con}_{01}([0^{+L})_L) \times \mathrm{Con}_{01}([0^{+M})_M) \oplus \mathcal{C}_2$.
- If $L \boxplus M \ncong \mathcal{C}_2^2$, $1 \in \mathrm{Sji}(L) \cap \mathrm{Sji}(M)$ and $0 \in \mathrm{Smi}(L) \cap \mathrm{Smi}(M)$, then:

$$\mathrm{Con}_{\mathbb{WDL}}(L \boxplus M, {}^{\Delta(L\boxplus M)\setminus\{1\}}, {}^{\nabla(L\boxplus M)\setminus\{0\}}) = \{(=_{\mathcal{C}_2} \oplus \varepsilon \oplus =_{\mathcal{C}_2}) \boxplus (=_{\mathcal{C}_2} \oplus \xi \oplus =_{\mathcal{C}_2}) \\ \mid \varepsilon \in \mathrm{Con}_{01}([0^{+L}, 1^{-L}]_L), \xi \in \mathrm{Con}_{01}([0^{+M}, 1^{-M}]_M)\} \cup \{(L \boxplus M)^2\} \\ \cong (\mathrm{Con}_{01}([0^{+L}, 1^{-L}]_L) \times \mathrm{Con}_{01}([0^{+M}, 1^{-M}]_M)) \oplus \mathcal{C}_2.$$

Example 6.1 Let us consider the five–element non–modular lattice $\mathcal{N}_5$, with the elements denoted as in the following leftmost Hasse diagram and notice that $\mathrm{Con}(\mathcal{N}_5) \cong \mathcal{C}_2 \oplus \mathcal{C}_2^2$ has the lattice structure represented in the second diagram below, in which the proper nontrivial congruences are indicated by their nonsingleton classes:



By Proposition 6.1, $\mathcal{N}_5 = \mathcal{C}_3 \boxplus \mathcal{C}_4$ has four weak dicomplementations: $\{({}^{\Delta}, {}^{\nabla}) \mid {}^{\Delta} \in \{{}^{\Delta\mathcal{N}_5}, {}^{\Delta\mathcal{N}_5\setminus\{1\}}\}, {}^{\nabla} \in \{{}^{\nabla\mathcal{N}_5}, {}^{\nabla\mathcal{N}_5\setminus\{0\}}\}\}$.

By Remark 6.1, $\mathrm{Con}_{\mathbb{WDL}}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5}, {}^{\nabla\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{WCL}}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{DWCL}}(\mathcal{N}_5, {}^{\nabla\mathcal{N}_5}) = \mathrm{Con}_{01}(\mathcal{N}_5) \cup \{\mathcal{N}_5^2\} = \{=_{\mathcal{C}_3} \boxplus =_{\mathcal{C}_4}, =_{\mathcal{C}_3} \boxplus (=_{\mathcal{C}_2} \oplus \mathcal{C}_2^2 \oplus =_{\mathcal{C}_2})\} \cup \{\mathcal{N}_5^2\} = \{=_{\mathcal{N}_5}, =_{\mathcal{C}_3} \boxplus (=_{\mathcal{C}_2} \oplus \mathcal{C}_2^2 \oplus =_{\mathcal{C}_2}), \mathcal{N}_5^2\} \cong \mathcal{C}_3$, represented in the third diagram above, since $\mathcal{C}_2 \cong \mathrm{Con}_{01}(\mathcal{N}_5) = \{=_{\mathcal{N}_5}, =_{\mathcal{C}_3} \boxplus (=_{\mathcal{C}_2} \oplus \mathcal{C}_2^2 \oplus =_{\mathcal{C}_2})\} = \{=_{\mathcal{N}_5}, \mathrm{eq}(\{0\}, \{a\}, \{b,c\}, \{1\})\} = \mathrm{Con}_{\mathbb{WDL}01}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5}, {}^{\nabla\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{WCL}01}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{DWCL}01}(\mathcal{N}_5, {}^{\nabla\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{WDL}0}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5}, {}^{\nabla\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{WCL}0}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{DWCL}0}(\mathcal{N}_5, {}^{\nabla\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{WDL}1}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5}, {}^{\nabla\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{WCL}1}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{DWCL}1}(\mathcal{N}_5, {}^{\nabla\mathcal{N}_5})$.

By Proposition 6.2, $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5\setminus\{1\}}) = \{=_{\mathcal{C}_3} \boxplus =_{\mathcal{C}_4}, \mathrm{eq}(\{0,b,c\}, \{a,1\}), \mathcal{N}_5^2\} = \{=_{\mathcal{N}_5}, \mathrm{eq}(\{0,b,c\}, \{a,1\}), \mathcal{N}_5^2\} \cong \mathcal{C}_3$, represented in the fourth diagram above, so $\mathrm{Con}_{\mathbb{WCL}01}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5\setminus\{1\}}) = \mathrm{Con}_{\mathbb{WCL}0}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5\setminus\{1\}}) = \mathrm{Con}_{\mathbb{WCL}1}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5\setminus\{1\}}) = \{=_{\mathcal{N}_5}\} \cong \mathcal{C}_1$ and $\mathrm{Con}_{\mathbb{DWCL}}(\mathcal{N}_5, {}^{\nabla\mathcal{N}_5\setminus\{0\}}) = \{=_{\mathcal{C}_3} \boxplus =_{\mathcal{C}_4}, \mathrm{eq}(\{0,a\}, \{b,c,1\}), \mathcal{N}_5^2\} = \{=_{\mathcal{N}_5}, \mathrm{eq}(\{0,a\}, \{b,c,1\}), \mathcal{N}_5^2\} \cong \mathcal{C}_3$, represented in the fifth diagram above, so $\mathrm{Con}_{\mathbb{DWCL}01}(\mathcal{N}_5, {}^{\nabla\mathcal{N}_5\setminus\{0\}}) = \mathrm{Con}_{\mathbb{DWCL}0}(\mathcal{N}_5, {}^{\nabla\mathcal{N}_5\setminus\{0\}}) = \mathrm{Con}_{\mathbb{DWCL}1}(\mathcal{N}_5, {}^{\nabla\mathcal{N}_5\setminus\{0\}}) = \{=_{\mathcal{N}_5}\} \cong \mathcal{C}_1$.

Consequently: $\mathrm{Con}_{\mathbb{WDL}}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5}, {}^{\nabla\mathcal{N}_5\setminus\{0\}}) = \mathrm{Con}_{\mathbb{WDL}}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5\setminus\{1\}}, {}^{\nabla\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{WDL}}(\mathcal{N}_5, {}^{\Delta\mathcal{N}_5\setminus\{1\}}, {}^{\nabla\mathcal{N}_5\setminus\{0\}}) = \{=_{\mathcal{N}_5}, \mathcal{N}_5^2\} \cong \mathcal{C}_2$, and, w.r.t. any of these nontrivial weak dicomplementations $({}^{\Delta}, {}^{\nabla})$, $\mathrm{Con}_{\mathbb{WDL}01}(\mathcal{N}_5, {}^{\Delta}, {}^{\nabla}) = \mathrm{Con}_{\mathbb{WDL}0}(\mathcal{N}_5, {}^{\Delta}, {}^{\nabla}) = \mathrm{Con}_{\mathbb{WDL}1}(\mathcal{N}_5, {}^{\Delta}, {}^{\nabla}) = \{=_{\mathcal{N}_5}\} \cong \mathcal{C}_1$.

Similar calculations give us these congruence lattices for the general case of the horizontal sum $\mathcal{C}_r \boxplus \mathcal{C}_s$ for any $r, s \in \mathbb{N}$ such that $r \geq 3$ and $s \geq 4$.

7 The Largest Numbers of Congruences of Finite Weakly Complemented Lattices

In the rest of this paper we present an application for the small investigation above on (dual) weak complementations that can be defined on bounded lattices with different structures and the lattice congruences that preserve them, which consists of determining the several largest numbers of congruences of an n–element (dual) weakly complemented lattice and those of an n–element weakly dicomplemented lattice, for n an arbitrary nonzero natural number. We proceed by induction on n, using the next lemma and some helpful results on lattices: Lemma 7.2 and Theorem 7.1. To be able to use the next lemma in an inductive argument along the lines of the ones from [3, 4, 19, 20] and obtain numbers of congruences which are small enough to occur w.r.t. nontrivial weak complementations, we must first investigate the atoms of the lattices of congruences of finite (dual) weakly complemented lattices that collapse at most three elements with other elements, specifically to determine the shapes of their nonsingleton congruence classes and the possible definitions of the weak complementation on the elements from these classes; we make this investigation in Lemma 7.3.

Lemma 7.1 [20] *If A is a congruence–distributive algebra and α is an atom of the lattice of congruences of A, then A has at most twice as many congruences as A/α.*

Let A be a member of a variety $\mathbb{V}$ of lattice–ordered algebras. Then, for any $S \subseteq A$, we have $S \subseteq Cg_A(S) \subseteq Cg_{\mathbb{V},A}(S)$, hence $Cg_{\mathbb{V},A}(S) = Cg_{\mathbb{V},A}(Cg_A(S))$. By the convexity of the congruence classes of lattice congruences, any principal congruence of a lattice and thus any principal congruence of A is generated by a pair of elements a, b with $a \leq b$. Thus, clearly, if A is finite, then any join–irreducible congruence of A and in particular any atom of the lattice of congruences of A is generated by a pair of elements $a, b \in A$ with $a \prec b$.

Remark 7.1 Let $(L, ^{\Delta})$ be a weakly complemented lattice.

By the above, if L is finite and $\alpha \in \mathrm{At}(\mathrm{Con}_{\mathbb{WCL}}(L))$, then, for some $a, b \in L$ with $a \prec b$, we have $\alpha = Cg_{\mathbb{WCL},L}(a, b) = Cg_{\mathbb{WCL},L}(Cg_L(a, b))$.

Let us also note that, for any $a \in L$, no proper congruence θ of $(L, ^{\Delta})$ can have a and a^{Δ} in the same class, because then we would have $a/\theta = a^{\Delta}/\theta = (a \vee a^{\Delta})/\theta = 1/\theta$, hence $1/\theta = a^{\Delta}/\theta = 1^{\Delta}/\theta = 0/\theta$.

For any set M and any nonempty subset $S \subseteq M$, let us denote by $\varepsilon_M(S)$ the equivalence on M having S as a class and all other classes singletons: $\varepsilon_M(S) = \mathrm{eq}(\{S\} \cup \{\{x\} : x \in M \setminus S\})$. Clearly, for any nonempty family $(S_i)_{i \in I}$ of parwise disjoint nonempty subsets of M, we have, in the lattice $\mathrm{Eq}(M)$: $\bigvee_{i \in I} \varepsilon_M(S_i) = \mathrm{eq}(\{S_i : i \in I\} \cup \{\{x\} : x \in M \setminus \bigcup_{i \in I} S_i\}) = \bigcup_{i \in I} \varepsilon_M(S_i)$. For brevity, if $a_1, \ldots, a_k \in M$ for some $k \in \mathbb{N}^*$, then we denote by $\varepsilon_M(a_1, \ldots, a_k) = \varepsilon_M(\{a_1, \ldots, a_k\})$.

For statement (iii) of the following lemma, see [19, Lemma 3.3 and Remark 3.4].

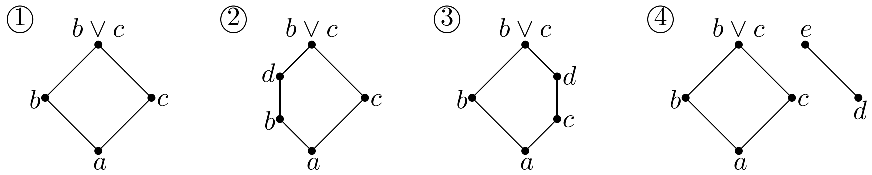
Lemma 7.2 [3, 11, 13, 19] *If L is a finite lattice and $a, b \in L$ are such that $a \prec b$:*

(i) $|L/Cg_L(a, b)| = |L| - 1$ *iff* $a \in \mathrm{Mi}(L)$ *and* $b \in \mathrm{Ji}(L)$ *iff* $Cg_L(a, b) = \varepsilon_L(a, b)$;

(ii) $|L/Cg_L(a, b)| = |L| - 2$ *iff the following or its dual (by dual meaning the case when b has a lower cover c such that in L^d the following hold for the interval $[a \wedge c, b]_L$ instead of $[a, b \vee c]_L$) holds: $a \notin \mathrm{Mi}(L)$ and, for some $c \in L$, $a \prec c$, $b \prec b \vee c$, $c \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, b \vee c\} \cong \mathcal{C}_2^2$ and $Cg_L(a, b) = \varepsilon_L(a, b) \cup \varepsilon_L(c, b \vee c)$;*

(iii) $|L/Cg_L(a, b)| = |L| - 3$ *iff the following or its dual (with the dual of the following being formulated as above) holds: $a \notin \mathrm{Mi}(L)$ so, for some $c \in L \setminus \{b\}$, we have $a \prec c$, and we are in one of the following situations, depicted in the following diagrams:*

① $b \prec b \vee c$, $c \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, b \vee c\} \cong \mathcal{C}_2^2$ and $Cg_L(a,b) = \varepsilon_L([a, b \vee c]_L)$;
② $c \prec b \vee c$ and, for some $d \in L$, $b \prec d \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, d, b \vee c\} \cong \mathcal{N}_5$ and $Cg_L(a,b) = \varepsilon_L(a,b,d) \cup \varepsilon_L(c, b \vee c)$;
③ $b \prec b \vee c$ and, for some $d \in L$, $c \prec d \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, d, b \vee c\} \cong \mathcal{N}_5$ and $Cg_L(a,b) = \varepsilon_L(a,b) \cup \varepsilon_L(c, d, b \vee c)$;
④ $b \prec b \vee c$, $c \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, b \vee c\} \cong \mathcal{C}_2^2$ and, for some $d, e \in L \setminus \{a, b, c, b \vee c\}$ such that $d \prec e$, $Cg_L(a,b) = \varepsilon_L(a,b) \cup \varepsilon_L(c, b \vee c) \cup \varepsilon_L(d,e)$.



Recall from Section 3 that, for any bounded lattice L, ${}^{\Delta L}$ is the trivial weak complementation on L, and from Remark 4.1 that ${}^{\Delta L}$ is the unique weak complementation on L if $1 \in \mathrm{Sji}(L)$, in particular if L is a chain.

Recall also from Section 5 that, if L is coatomic and has exactly two coatoms, then ${}^{\Delta L \setminus \{1\}}$ is nontrivial; furthermore, by Lemma 4.1 and Proposition 6.1, for any bounded lattices K, L, M such that $|L|, |M| > 3$:

• if $1 \notin \mathrm{Sji}(L) \cap \mathrm{Sji}(M)$, then $K \oplus (L \boxplus M)$ can only be endowed with the trivial weak complementation ${}^{\Delta K \oplus (L \boxplus M)}$;

• if $1 \in \mathrm{Sji}(L) \cap \mathrm{Sji}(M)$, then the only nontrivial weak complementation on $K \oplus (L \boxplus M)$ is ${}^{\Delta (K \oplus (L \boxplus M)) \setminus \{1\}}$.

Thus, for any bounded lattice K, ${}^{\Delta (K \oplus \mathcal{C}_2^2) \setminus \{1\}}$ is the only nontrivial weak complementation on $K \oplus \mathcal{C}_2^2$, which restricts to the Boolean complementation on $\mathcal{C}_2^2$, and ${}^{\Delta (K \oplus \mathcal{N}_5) \setminus \{1\}}$ is the only nontrivial weak complementation on $K \oplus \mathcal{N}_5$.

Lemma 7.3 *For any finite weakly complemented lattice* $(L, {}^{\Delta})$ *and any* $a, b \in L$ *such that* $a \prec b$, *we have*:

i. $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L| - 1$ *iff* $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b) = \varepsilon_L(a,b)$ *iff* $a \in \mathrm{Mi}(L)$, $b \in \mathrm{Ji}(L)$ *and either* $L = \{a, b\} \cong \mathcal{C}_2$ *or* $a^{\Delta} = b^{\Delta} = 1$;

ii. $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L| - 2$ *iff we are in one of the following cases*:

ⓐ $b = 1$ and $L = \{0, a, 1\} \cong \mathcal{C}_3$, in particular $Cg_L(a,b) = \varepsilon_L(a,b) \subsetneq Cg_{\mathbb{WCL},L}(a,b)$;
ⓑ $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b)$, $Cg_L(a,b)$ is as in Lemma 7.2, (ii), and one of the following holds:

(β_1) $a^{\Delta} = b^{\Delta}$ and, in the case when $a \prec c \in L \setminus \{b\}$, $c^{\Delta} = (b \vee c)^{\Delta}$, while, in the lattice dual of this situation, $c^{\Delta} = (a \wedge c)^{\Delta}$;
(β_2) in the case when $a \prec c \in L \setminus \{b\}$, $L = \{a, b, c, b \vee c\} \cong \mathcal{C}_2^2$ and ${}^{\Delta} = {}^{\Delta L \setminus \{1\}}$, and, in the lattice dual of this situation, $L = \{a \wedge c, c, a, b\} \cong \mathcal{C}_2^2$ and ${}^{\Delta} = {}^{\Delta L \setminus \{1\}}$;

iii. $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L| - 3$ *iff we are in one of the following cases or their lattice duals*:

ⓖ $a \in \mathrm{Mi}(L), b \in \mathrm{Ji}(L), Cg_L(a,b) = \varepsilon_L(a,b)$ and $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L/Cg_L(a,b)| - 2$, case in which $L \cong \mathcal{C}_4$, $b = 1$ and a is the coatom of L;
ⓓ $a \prec c$ for some $c \in L \setminus \{a\}$ such that $b \prec b \vee c$, $c \prec b \vee c$ and $[a, b \vee c]_L = \{a, b, c, b \vee c\} \cong \mathcal{C}_2^2$, $Cg_L(a,b) = \varepsilon_L(a,b) \cup \varepsilon_L(c, b \vee c)$, and $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L/Cg_L(a,b)| - 1$, case in which we are in one of the following subcases:

- (δ_1) $a = 0$ and $b \vee c = 1$, so $L = \{a, b, c, b \vee c\} \cong \mathcal{C}_2^2$, and ${}^\Delta = {}^{\Delta L}$, so $Cg_{\mathbb{WCL},L}(a,b) = L^2$;
- (δ_2) $0 \prec a$, $b \vee c = 1$, $L = \{0, a, b, c, 1\} \cong \mathcal{C}_2 \oplus \mathcal{C}_2^2$, ${}^\Delta = {}^{\Delta L \setminus \{1\}}$ and $Cg_{\mathbb{WCL},L}(a,b) = \varepsilon_L(0, a, b) \cup \varepsilon_L(c, 1)$;
- (δ_3) $(b \vee c)^\Delta \prec c^\Delta < b^\Delta = a^\Delta$, in particular ${}^\Delta$ has at least three distinct values and thus it is nontrivial, $Cg_{\mathbb{WCL},L}(a,b) = \varepsilon_L(a,b) \cup \varepsilon_L(c, b \vee c) \cup \varepsilon_L(c^\Delta, (b \vee c)^\Delta)$, $a^{\Delta\Delta} = b^{\Delta\Delta} = a$, $c^{\Delta\Delta} = c$ and $(b \vee c)^{\Delta\Delta} = b \vee c$;
- (δ_4) $(b \vee c)^\Delta = b^\Delta \prec a^\Delta = c^\Delta$, in particular $b \vee c \neq 1$, so ${}^\Delta$ has at least three distinct values and thus it is nontrivial, $Cg_{\mathbb{WCL},L}(a,b) = \varepsilon_L(a,b) \cup \varepsilon_L(c, b \vee c) \cup \varepsilon_L(a^\Delta, b^\Delta)$, $a^{\Delta\Delta} = c^{\Delta\Delta} = a$ and $b^{\Delta\Delta} = (b \vee c)^{\Delta\Delta} = b$;

ⓔ $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b)$ and $Cg_L(a,b)$ is as in case ① in Lemma 7.2, (iii), case in which one of the following holds:

- (ϵ_1) $a^\Delta = b^\Delta = c^\Delta = (b \vee c)^\Delta$;
- (ϵ_2) $L = \{a, b, c, b \vee c\} \cong \mathcal{C}_2^2$ and ${}^\Delta = {}^{\Delta L \setminus \{1\}}$;

ⓕ $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b)$ and $Cg_L(a,b)$ is as in case ② in Lemma 7.2, (iii), case in which one of the following holds:

- (φ_1) $a^\Delta = b^\Delta = d^\Delta$ and $c^\Delta = (b \vee c)^\Delta$, in particular $b \vee c \neq 1$;
- (φ_2) $L = \{a, b, c, d, b \vee c\} \cong \mathcal{N}_5$ and ${}^\Delta = {}^{\Delta L \setminus \{1\}}$;

ⓨ $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b)$ and $Cg_L(a,b)$ is as in case ③ in Lemma 7.2, (iii), case in which one of the following holds:

- (ψ_1) $a^\Delta = b^\Delta$ and $c^\Delta = d^\Delta = (b \vee c)^\Delta$, in particular $b \vee c \neq 1$;
- (ψ_2) $L = \{a, b, c, d, b \vee c\} \cong \mathcal{N}_5$ and ${}^\Delta = {}^{\Delta L \setminus \{1\}}$;

ⓧ $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b)$ and $Cg_L(a,b)$ is as in case ④ in Lemma 7.2, (iii), case in which one of the following holds:

- (χ_1) $a^\Delta = b^\Delta$, $c^\Delta = (b \vee c)^\Delta$ and $d^\Delta = e^\Delta$, in particular $b \vee c \neq 1$ and $e \neq 1$;
- (χ_2) ${}^\Delta$ is nontrivial and $\mathcal{C}_4 \boxplus \mathcal{C}_4 \cong \{c \wedge d, c, d, e, b \vee c, 1\}$ is a sublattice of L;
- (χ_3) $a = 0$, $e = 1$, $\mathcal{C}_2 \times \mathcal{C}_3 \cong \{a, b, c, b \vee c, d, e\}$ is a bounded sublattice of L, $b^\Delta = d$ and $d^\Delta = b$;
- (χ_4) $d = 0$, $b \vee c = 1$, $\mathcal{C}_2 \times \mathcal{C}_3 \cong \{d, e, a, b, c, b \vee c\}$ is a bounded sublattice of L, $e^\Delta = c$ and $c^\Delta = e$.

Proof We will repeatedly use Remark 7.1.

(i) By Lemma 7.2, (i), $a \in \mathrm{Mi}(L)$ and $b \in \mathrm{Ji}(L)$. But $b = (b \wedge a) \vee (b \wedge a^\Delta) = a \vee (b \wedge a^\Delta)$, and $a < b$, hence $b \wedge a^\Delta = b$, so $a^\Delta \geq b > a$, thus, since $a \vee a^\Delta = 1$, it follows that $a^\Delta = 1$, and, since $Cg_{\mathbb{WCL},L}(a,b)$ only collapses a with b, either $b^\Delta = a^\Delta = 1$ or $b^\Delta \in \{a, b\}$, case in which $Cg_{\mathbb{WCL},L}(a,b) = L^2$, thus $|L| = |L/Cg_{\mathbb{WCL},L}(a,b)| + 1 = 1 + 1 = 2$, so $L = \{a, b\} \cong \mathcal{C}_2$.

(ii) The fact that $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L| - 2$ implies that we are in one of the following cases:

ⓐ $a \in \mathrm{Mi}(L)$ and $b \in \mathrm{Ji}(L)$, so that $Cg_L(a,b) = \varepsilon_L(a,b)$ by Lemma 7.2, (i), and $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L/Cg_L(a,b)| - 1$, in particular $Cg_L(a,b) \subsetneq Cg_{\mathbb{WCL},L}(a,b)$;

(β) $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b)$, which is as in Lemma 7.2, (ii), and below we will consider the case when $a \prec c \in L \setminus \{b\}$, with its dual being treated similarly.

We can not have $a \in \mathrm{Mi}(L)$, $b \in \mathrm{Ji}(L)$ and $a^\Delta = b^\Delta$, because then $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b) = \varepsilon_L(a,b)$, thus $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L| - 1$, which would contradict the current hypothesis.

Hence, in the case (α), where $a \in \mathrm{Mi}(L)$ and $b \in \mathrm{Ji}(L)$, so that $Cg_L(a,b) = \varepsilon_L(a,b)$, we have $a^\Delta \neq b^\Delta$, thus also $a^{\Delta\Delta} \neq b^{\Delta\Delta}$, since otherwise we would get $a^\Delta = a^{\Delta\Delta\Delta} = b^{\Delta\Delta\Delta} = b^\Delta$. Of course, since $(a,b) \in Cg_{\mathbb{WCL},L}(a,b)$, we also have $(a^\Delta, b^\Delta), (a^{\Delta\Delta}, b^{\Delta\Delta}) \in Cg_{\mathbb{WCL},L}(a,b)$, and, since in this case $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L/Cg_L(a,b)| - 1$, either at most one of the elements $a^\Delta, b^\Delta, a^{\Delta\Delta}, b^{\Delta\Delta}$ does not belong to $\{a,b\} = a/Cg_L(a,b) = b/Cg_L(a,b) \subseteq a/Cg_{\mathbb{WCL},L}(a,b) = b/Cg_{\mathbb{WCL},L}(a,b)$, so that a^Δ or b^Δ belongs to $a/Cg_{\mathbb{WCL},L}(a,b) = b/Cg_{\mathbb{WCL},L}(a,b)$, or $a^\Delta/Cg_{\mathbb{WCL},L}(a,b) = \{a^\Delta, b^\Delta\} = \{a^{\Delta\Delta}, b^{\Delta\Delta}\}$; in either of these subcases, we have $x^\Delta \in x/Cg_{\mathbb{WCL},L}(a,b)$ for some $x \in L$, hence $Cg_{\mathbb{WCL},L}(a,b) = L^2$, so that $|L/Cg_{\mathbb{WCL},L}(a,b)| = 1$, thus $|L| = |L/Cg_{\mathbb{WCL},L}(a,b)| + 2 = 3$, therefore $L \cong \mathcal{C}_3$, so $^\Delta$ is the trivial weak complementation, thus, since $a^\Delta \neq b^\Delta$ and $a \prec b$, it follows that $b = 1$ and a is the single element of $L \setminus \{0,1\}$.

In case (β), since $Cg_{\mathbb{WCL},L}(a,b)$ collapses no other elements but a with b and c with $b \vee c$:

either $a^\Delta = b^\Delta$ and $c^\Delta = (b \vee c)^\Delta$,

or $x^\Delta \in \{a,b,c,b \vee c\}$ for some $x \in \{a,b,c,b \vee c\}$, but then, since $x \vee x^\Delta = 1$ and $b \vee c = \max\{a,b,c,b \vee c\}$, it follows that $b \vee c = 1$, thus $c/Cg_{\mathbb{WCL},L}(a,b) = 1/Cg_{\mathbb{WCL},L}(a,b)$, hence $c^\Delta/Cg_{\mathbb{WCL},L}(a,b) = 0/Cg_{\mathbb{WCL},L}(a,b)$, but, since $c \neq 1$ and thus $c^\Delta \neq 0$, we have $0, c^\Delta \in \{a,b,c,b \vee c\}$, hence $a = 0$ and $c^\Delta = b$ since $a = \min\{a,b,c,b \vee c\}$, therefore, by Lemma 7.2, (ii), $L = [0,1]_L = [a, b \vee c]_L = \{a,b,c,b \vee c\} \cong \mathcal{C}_2^2$ and $^\Delta = ^{\Delta L \setminus \{1\}}$.

(iii) $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L| - 3$ iff one of the following holds:

(γ) $|L/Cg_L(a,b)| = |L| - 1$ and $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L/Cg_L(a,b)| - 2$, so that $Cg_L(a,b)$ is as in Lemma 7.2, (i), and $Cg_L(a,b) \subsetneq Cg_{\mathbb{WCL},L}(a,b)$;

(δ) $|L/Cg_L(a,b)| = |L| - 2$ and $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L/Cg_L(a,b)| - 1$, so that $Cg_L(a,b)$ is as in Lemma 7.2, (ii), and $Cg_L(a,b) \subsetneq Cg_{\mathbb{WCL},L}(a,b)$;

(ζ) $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L/Cg_L(a,b)| = |L| - 3$, so that $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b)$, which is as in Lemma 7.2, (iii).

(γ) In this subcase, by Lemma 7.2, (i), $a \in \mathrm{Mi}(L)$, $b \in \mathrm{Ji}(L)$ and $Cg_L(a,b) = \varepsilon_L(a,b)$, and, since $Cg_L(a,b) \subsetneq Cg_{\mathbb{WCL},L}(a,b)$, we have $a^\Delta \neq b^\Delta$, thus also $a^{\Delta\Delta} \neq b^{\Delta\Delta}$.

Since $(a,b) \in Cg_{\mathbb{WCL},L}(a,b)$, we also have $(a^\Delta, b^\Delta), (a^{\Delta\Delta}, b^{\Delta\Delta}) \in Cg_{\mathbb{WCL},L}(a,b)$, thus $Cg_{\mathbb{WCL},L}(a,b) \supseteq Cg_L(a,b) \vee Cg_L(a^\Delta, b^\Delta) \vee Cg_L(a^{\Delta\Delta}, b^{\Delta\Delta}) = \varepsilon_L(a,b) \vee Cg_L(a^\Delta, b^\Delta) \vee Cg_L (a^{\Delta\Delta}, b^{\Delta\Delta})$.

• If the sets $\{a,b\}$, $\{a^\Delta, b^\Delta\}$ and $\{a^{\Delta\Delta}, b^{\Delta\Delta}\}$ are pairwise disjoint, then $^\Delta$ is nontrivial, and the fact that $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L| - 3 \geq 3$ ensures us that $Cg_{\mathbb{WCL},L}(a,b) = \varepsilon_L(a,b) \cup \varepsilon_L(a^\Delta, b^\Delta) \cup \varepsilon_L(a^{\Delta\Delta}, b^{\Delta\Delta}) \subsetneq L^2$, hence $b^\Delta \prec a^\Delta$, $a^{\Delta\Delta} \prec b^{\Delta\Delta}$ and either $Cg_L(a^\Delta, b^\Delta) = \varepsilon_L(a^\Delta, b^\Delta)$ and $Cg_L(a^{\Delta\Delta}, b^{\Delta\Delta}) = \varepsilon_L(a^{\Delta\Delta}, b^{\Delta\Delta})$ or $Cg_L(a^\Delta, b^\Delta) = Cg_L(a^{\Delta\Delta}, b^{\Delta\Delta}) = \varepsilon_L(a^\Delta, b^\Delta) \cup \varepsilon_L(a^{\Delta\Delta}, b^{\Delta\Delta})$. Note also that $a^{\Delta\Delta} < a$ and $b^{\Delta\Delta} < b$ since $a^{\Delta\Delta} \leq a$, $b^{\Delta\Delta} \leq b$ and $\{a,b\} \cap \{a^{\Delta\Delta}, b^{\Delta\Delta}\} = \emptyset$.

If $Cg_L(a^\Delta, b^\Delta) = Cg_L(a^{\Delta\Delta}, b^{\Delta\Delta}) = \varepsilon_L(a^\Delta, b^\Delta) \cup \varepsilon_L(a^{\Delta\Delta}, b^{\Delta\Delta})$, then, by Lemma 7.2, (ii), $\{a^\Delta, b^\Delta, a^{\Delta\Delta}, b^{\Delta\Delta}\} \cong \mathcal{C}_2^2$, so that $1 = a^\Delta \vee a^{\Delta\Delta} \in \mathrm{Max}(\{a^\Delta, b^\Delta, a^{\Delta\Delta}, b^{\Delta\Delta}\}) \subseteq \{a^\Delta, b^{\Delta\Delta}\}$. Since $b^{\Delta\Delta} < b$, it follows that $b^{\Delta\Delta} \neq 1$, hence $a^\Delta = 1$ and, by Lemma 7.2, (ii), $a^{\Delta\Delta} \prec b^\Delta$ and $b^{\Delta\Delta} \prec a^\Delta = 1$; but $b^{\Delta\Delta} < b \leq 1 = a^\Delta$, thus $b = 1 = a^\Delta$, contradicting the fact that $\{a,b\}$ and $\{a^\Delta, b^\Delta\}$ are disjoint.

Hence $Cg_L(a^\Delta, b^\Delta) = \varepsilon_L(a^\Delta, b^\Delta)$ and $Cg_L(a^{\Delta\Delta}, b^{\Delta\Delta}) = \varepsilon_L(a^{\Delta\Delta}, b^{\Delta\Delta})$, so that, by Lemma 7.2, (i), $b^\Delta, a^{\Delta\Delta} \in \mathrm{Mi}(L)$ and $a^\Delta, b^{\Delta\Delta} \in \mathrm{Ji}(L)$. But $a^\Delta = (a^\Delta \wedge b^\Delta) \vee (a^\Delta \wedge b^{\Delta\Delta}) =$

$b^\Delta \vee (a^\Delta \wedge b^{\Delta\Delta})$, and, since $b^\Delta < a^\Delta \in \mathrm{Ji}(L)$, it follows that $a^\Delta = a^\Delta \wedge b^{\Delta\Delta}$, thus $a^\Delta \leq b^{\Delta\Delta} > a^{\Delta\Delta}$, hence $1 = a^\Delta \vee a^{\Delta\Delta} \leq b^{\Delta\Delta}$, thus $1 = b^{\Delta\Delta} < b$, a contradiction.

• If $\{a,b\} \cap \{a^\Delta, b^\Delta\} \neq \emptyset$ or $\{a^\Delta, b^\Delta\} \cap \{a^{\Delta\Delta}, b^{\Delta\Delta}\} \neq \emptyset$, then $Cg_{\mathbb{WCL},L}(a,b) = L^2$, thus $|L/Cg_{\mathbb{WCL},L}(a,b)| = 1$, hence $|L| = 1+3 = 4$, so $L \cong \mathcal{C}_2^2$ or $L \cong \mathcal{C}_4$, but $L \cong \mathcal{C}_2^2$ would contradict the fact that $a \prec b$, $a \in \mathrm{Mi}(L)$ and $b \in \mathrm{Ji}(L)$, thus $L \cong \mathcal{C}_4$, so $^\Delta = ^{\Delta L}$, and, since $a \prec b$ and $a^\Delta \neq b^\Delta$, it follows that $b = 1$ and a is the coatom of L.

• If $\{a,b\} \cap \{a^{\Delta\Delta}, b^{\Delta\Delta}\} \neq \emptyset$, then, since $a^{\Delta\Delta} < b^{\Delta\Delta}$, but also $a^{\Delta\Delta} \leq a$ and $b^{\Delta\Delta} \leq b$, it follows that $b^{\Delta\Delta} = a$, so, by the fact that $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L| - 3$, either, as above, $Cg_{\mathbb{WCL},L}(a,b) = L^2$, $L \cong \mathcal{C}_4$ and $b = 1$, or $\{a,b\} \cap \{a^\Delta, b^\Delta\} = \{a^\Delta, b^\Delta\} \cap \{a^{\Delta\Delta}, b^{\Delta\Delta}\} = \emptyset$, $a^{\Delta\Delta} \prec a = b^{\Delta\Delta}$ and $b^\Delta \prec a^\Delta$; however, the latter implies $b^{\Delta\Delta} \leq a$, hence $a^\Delta \leq b^\Delta$, contradicting $b^\Delta < a^\Delta$.

ⓓ In this subcase, by Lemma 7.2, (ii), the following or its lattice dual holds: $a \prec c$ for some $c \in L \setminus \{b\}$ such that $b \prec b \vee c$, $c \prec b \vee c$ and $Cg_L(a,b) = \varepsilon_L(a,b) \cup \varepsilon_L(c, b \vee c)$.

We can't have $a^\Delta = b^\Delta$ and $c^\Delta = (b \vee c)^\Delta$, because then we'd have $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b)$, contradicting the above.

■ If $x^\Delta \in x/Cg_{\mathbb{WCL},L}(a,b)$ for some $x \in \{a,b,c,b\vee c\}$, then $Cg_{\mathbb{WCL},L}(a,b) = L^2$, thus $|L| = |L/Cg_{\mathbb{WCL},L}(a,b)| + 3 = 1 + 3 = 4$, hence $L = \{a,b,c,b\vee c\} \cong \mathcal{C}_2^2$, with $a = 0$ and $b \vee c = 1$. The fact that $x^\Delta \in x/Cg_{\mathbb{WCL},L}(a,b)$ for some $x \in \{a,b,c,b\vee c\}$ implies that $^\Delta \neq ^{\Delta b,c}$, hence $^\Delta = ^{\Delta L}$ (so that $c^\Delta = 1 \in c/Cg_{\mathbb{WCL},L}(a,b)$).

■ Now let us assume that no $x \in \{a,b,c,b\vee c\}$ has $x^\Delta \in x/Cg_{\mathbb{WCL},L}(a,b)$.

▶ If $x^\Delta \in \{a,b,c,b\vee c\}$ for some $x \in \{a,b,c,b\vee c\}$, then $b \vee c = 1$, hence $0 = (b\vee c)^\Delta \in c^\Delta/Cg_{\mathbb{WCL},L}(a,b)$. If $a = 0$, so that $L = [0,1]_L = \{a,b,c,b\vee c\}$, then $c^\Delta = b$, so $c^\Delta \leq b$, hence $b^\Delta \leq c$, thus $b^\Delta = c$, so $^\Delta = ^{\Delta b,c}$, but then $Cg_{\mathbb{WCL},L}(a,b) = Cg_L(a,b)$, contradicting the above. Hence $a \neq 0$, thus $0 \notin \{a,b,c,b\vee c\}$.

• If $c^\Delta \in \{a,b,c,b\vee c\} \setminus c/Cg_{\mathbb{WCL},L}(a,b) = \{a,b\}$, then $c^\Delta \leq b$, thus $b^\Delta \leq c$, and the fact that $c^\Delta/Cg_{\mathbb{WCL},L}(a,b) = 0/Cg_{\mathbb{WCL},L}(a,b)$ and $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L| - 3$ ensures us that $Cg_{\mathbb{WCL},L}(a,b) = \varepsilon_L(0,a,b) \cup \varepsilon_L(c, b\vee c) = \varepsilon_L(0,a,b) \cup \varepsilon_L(c,1)$, so $0 \prec a$; then $c^\Delta \neq a$, because otherwise $c^\Delta \leq a$, thus $a^\Delta \leq c$, so $a \vee a^\Delta \leq c < 1$, a contradiction; thus $c^\Delta = b$, hence $b^\Delta = c$, because $b^\Delta < c$ would imply $c^\Delta \leq b^\Delta < c$, a contradiction. If there existed any $x \in L \setminus \{0,a,b,c,b\vee c\} = L \setminus \{0,a,b,c,1\}$, then we can't have $x > a$, because then $x \in \mathrm{CoAt}(L) \setminus \{b,c\}$, so that $\{a,b,c,x,1\} \cong \mathcal{M}_3$ is a sublattice of L, which would contradict the fact that $(b,c) \notin Cg_L(a,b)$ since $\mathcal{M}_3$ is a simple lattice. Thus there exists a $y \in \mathrm{At}(L) \setminus \{a\}$ and $\mathrm{CoAt}(L) = \{b,c\}$, therefore either $y < b$, case in which we'd get the contradiction $y \in b/Cg_L(a,b)$, or $y < c$, case in which we'd get the contradiction $y \in c/Cg_L(a,b)$. Therefore, in this case, $L = \{0,a,b,c,b\vee c\} = \{0,a,b,c,1\} \cong \mathcal{C}_2 \oplus \mathcal{C}_2^2$ and $^\Delta = ^{\Delta b,c}$.

• If $c^\Delta \notin \{a,b,c,b\vee c\}$, then the fact that $|L/Cg_{\mathbb{WCL},L}(a,b)| = |L| - 3$ implies $0 \prec c^\Delta$ and $Cg_{\mathbb{WCL},L}(a,b) = \varepsilon_L(0,c^\Delta) \cup \varepsilon_L(a,b) \cup \varepsilon_L(b, b\vee c)$. Thus $c^{\Delta\Delta} \in 0^\Delta/Cg_{\mathbb{WCL},L}(a,b) = 1/Cg_{\mathbb{WCL},L}(a,b) = \{c,1\}$ and $c^{\Delta\Delta} \leq c$, so $c^{\Delta\Delta} = c$. Also, by the same argument as above, the atom c^Δ can not be comparable to either of b and c, thus $c^\Delta \vee b = 1$ and $c > c \wedge c^\Delta$, thus $c \wedge c^\Delta = 0$. But then L has the bounded sublattice $\{0,c^\Delta,c,1\} = \{0,c^\Delta,c,b\vee c\} \cong \mathcal{C}_2^2$, so that $(0,c^\Delta) \in Cg_L(c, b\vee c) = Cg_L(a,b) = \varepsilon_L(a,b) \cup \varepsilon_L(c, b\vee c)$, and we have a contradiction.

▶ The remaining subcase is $x^\Delta \notin \{a,b,c,b\vee c\}$ for any $x\in\{a,b,c,b\vee c\}$. But $(a,b),(c,b\vee c)\in Cg_L(a,b)\subsetneq Cg_{\mathbb{WCL},L}(a,b)$, thus $(a^\Delta,b^\Delta),(c^\Delta,(b\vee c)^\Delta)\in Cg_{\mathbb{WCL},L}(a,b)$, and $a^\Delta\neq b^\Delta$ or $c^\Delta\neq(b\vee c)^\Delta$, thus $Cg_{\mathbb{WCL},L}(a,b)=\varepsilon_L(a,b)\cup \varepsilon_L(c,b\vee c)\cup\varepsilon_L(C)$, where:

- if $(a^\Delta,c^\Delta)\notin Cg_{\mathbb{WCL},L}(a,b)$, then one of the sets $\{a^\Delta,b^\Delta\}$ and $\{c^\Delta,(b\vee c)^\Delta\}$ is a singleton and the other one is a two–element class C of $Cg_{\mathbb{WCL},L}(a,b)$;
  - ∗ if $a^\Delta\neq b^\Delta$, then $c^\Delta=(b\vee c)^\Delta$ and $a^{\Delta\Delta}\neq b^{\Delta\Delta}$, so $a^{\Delta\Delta},b^{\Delta\Delta}$ belong to a nonsingleton class of $Cg_{\mathbb{WCL},L}(a,b)$, which in the current subcase, since $a^{\Delta\Delta}\leq a$ and $b^{\Delta\Delta}\leq b$, can only be $\{a,b\}$, so $a^{\Delta\Delta}=a<b=b^{\Delta\Delta}$; also, $b^\Delta\neq a^\Delta=(b\wedge c)^\Delta=b^\Delta\vee c^\Delta$, thus $c^\Delta\nleq b^\Delta$, a contradiction;
  - · if $a^\Delta=b^\Delta$, then $c^\Delta\neq(b\vee c)^\Delta$, so $b^\Delta=a^\Delta=(b\wedge c)^\Delta=b^\Delta\vee c^\Delta$ and $c^{\Delta\Delta}\neq(b\vee c)^{\Delta\Delta}$, thus $(b\vee c)^\Delta<c^\Delta\leq b^\Delta=a^\Delta$; but $(a^\Delta,c^\Delta)\notin Cg_{\mathbb{WCL},L}(a,b)$, so $a^\Delta\neq c^\Delta$, thus $(b\vee c)^\Delta\prec c^\Delta<b^\Delta=a^\Delta$ since here $C=\{c^\Delta,(b\vee c)^\Delta\}$ is a class of $Cg_{\mathbb{WCL},L}(a,b)$, hence $(b\vee c)^{\Delta\Delta}>c^{\Delta\Delta}>b^{\Delta\Delta}=a^{\Delta\Delta}$ and $a^{\Delta\Delta}=b^{\Delta\Delta},c^{\Delta\Delta},(b\vee c)^{\Delta\Delta}\in\{a,b,c,b\vee c\}\cong\mathcal{C}_2^2$, thus, since $c^{\Delta\Delta}\leq c$, we have $a^{\Delta\Delta}=b^{\Delta\Delta}=a$, $c^{\Delta\Delta}=c$ and $(b\vee c)^{\Delta\Delta}=b\vee c$;
- if $(a^\Delta,c^\Delta)\in Cg_{\mathbb{WCL},L}(a,b)$, then $C=\{a^\Delta,b^\Delta,c^\Delta,(b\vee c)^\Delta\}$ is a two–element class of $Cg_{\mathbb{WCL},L}(a,b)$, thus $(b\vee c)^\Delta<a^\Delta=(b\wedge c)^\Delta=b^\Delta\vee c^\Delta$; as in subcase ∗ above, we can not have $a^\Delta\neq b^\Delta$ and $c^\Delta=(b\vee c)^\Delta$, thus: $a^\Delta=b^\Delta$ iff $c^\Delta=(b\vee c)^\Delta$;
  - · if $a^\Delta=c^\Delta$, then, by the above, $b^\Delta\neq a^\Delta=c^\Delta\neq(b\vee c)^\Delta$, so $b^\Delta=(b\vee c)^\Delta\prec a^\Delta=c^\Delta$ since $b^\Delta\leq a^\Delta$ and $\{a^\Delta,b^\Delta,c^\Delta,(b\vee c)^\Delta\}$ is a two–element class of $Cg_{\mathbb{WCL},L}(a,b)$, hence $a^{\Delta\Delta}=c^{\Delta\Delta}<b^{\Delta\Delta}=(b\vee c)^{\Delta\Delta}$, but, since $(a,b)\in Cg_{\mathbb{WCL},L}(a,b)$, $a^{\Delta\Delta}$ and $b^{\Delta\Delta}$ must belong to a nonsingleton class of $Cg_{\mathbb{WCL},L}(a,b)$, that in the current subcase can only be $\{a,b\}$, thus $a^{\Delta\Delta}=c^{\Delta\Delta}=a\prec b=b^{\Delta\Delta}=(b\vee c)^{\Delta\Delta}$;
  - · if $a^\Delta\neq c^\Delta$, then, by the above, $a^\Delta=b^\Delta$ and $c^\Delta=(b\vee c)^\Delta$, which implies $Cg_L(a,b)=\varepsilon_L(a,b)\cup\varepsilon_L(c,b\vee c)=Cg_{\mathbb{WCL},L}(a,b)$, a contradiction.

ⓩ In this subcase, we treat separately the subcases of Lemma 7.2, (iii), using the notations from this lemma.
(ϵ) If $Cg_L(a,b)$ is as in case ① in Lemma 7.2, (iii), then $\{a,b,c,b\vee c\}$ is the only nonsingleton class of $Cg_{\mathbb{WCL},L}(a,b)$, so we have either $a^\Delta=b^\Delta=c^\Delta=(b\vee c)^\Delta$ or $x^\Delta\in\{a,b,c,b\vee c\}$ for some $x\in\{a,b,c,b\vee c\}$, which implies $Cg_{\mathbb{WCL},L}(a,b)=L^2$, so that $|L/Cg_{\mathbb{WCL},L}(a,b)|=1$, thus $|L|=|L/Cg_{\mathbb{WCL},L}(a,b)|+3=4$, hence $L=\{a,b,c,b\vee c\}\cong\mathcal{C}_2^2$.
(φ) If $Cg_L(a,b)$ is as in case ② in Lemma 7.2, (iii), then $\{a,b,d\}$ and $\{c,b\vee c\}$ are the only nonsingleton classes of $Cg_{\mathbb{WCL},L}(a,b)$, hence either $a^\Delta=b^\Delta=d^\Delta$ and $c^\Delta=(b\vee c)^\Delta$ or $x^\Delta\in\{a,b,c,d,b\vee c\}$ for some $x\in\{a,b,c,d,b\vee c\}$, so that $1=x\vee x^\Delta\in\{a,b,c,d,b\vee c\}$, thus $b\vee c=1$, so $c/Cg_{\mathbb{WCL},L}(a,b)=1/Cg_{\mathbb{WCL},L}(a,b)$, hence $c^\Delta/Cg_{\mathbb{WCL},L}(a,b)=0/Cg_{\mathbb{WCL},L}(a,b)$ and $c^\Delta\neq 0$ since $c\neq 1$, therefore $0,c^\Delta\in\{a,b,c,d,b\vee c\}$, thus $a=0$, hence $L=[0,1]_L=[a,b\vee c]_L=\{a,b,c,d,b\vee c\}\cong\mathcal{N}_5$ and $^\Delta$ is the only nontrivial weak complementation on $\mathcal{N}_5$, namely $^{\Delta c,d}$ here.
(ψ) Analogously for the situation when $Cg_L(a,b)$ is as in case ③ in Lemma 7.2, (iii).
(χ) If $Cg_L(a,b)$ is as in case ④ in Lemma 7.2, (iii), so $Cg_{\mathbb{WCL},L}(a,b)=Cg_L(a,b)=\varepsilon_L(a,b)\cup\varepsilon_L(c,b\vee c)\cup\varepsilon_L(d,e)\neq L^2$, hence $x^\Delta\notin x/Cg_{\mathbb{WCL},L}(a,b)$ for any $x\in L$; then we are in one of the following subcases:

■ $x^\Delta \notin \{a,b,c,b\vee c,d,e\}$ for any $x\in\{a,b,c,b\vee c,d,e\}$; then, since $Cg_{\mathbb{WCL},L}(a,b)$ collapses only a with b, c with $b\vee c$ and d with e, it follows that $a^\Delta=b^\Delta$, $c^\Delta=(b\vee c)^\Delta$ and $d^\Delta=e^\Delta$;

■ $x^\Delta\in\{a,b,c,b\vee c,d,e\}$ for some $x\in\{a,b,c,b\vee c,d,e\}$ and one of the following holds: $a^\Delta\neq b^\Delta$, $c^\Delta\neq(b\vee c)^\Delta$ or $d^\Delta\neq e^\Delta$; then, since $x\vee x^\Delta=1$ and $\mathrm{Max}\{a,b,c,b\vee c,d,e\}\subseteq\{b\vee c,e\}$, we have $b\vee c\vee e=1$; also, $\{a^\Delta,b^\Delta,c^\Delta,(b\vee c)^\Delta,d^\Delta,e^\Delta\}\nsubseteq\{1\}$, thus, since $^\Delta$ is order–reversing, we have $(b\vee c)^\Delta<1$ or $e^\Delta<1$, in particular $^\Delta$ is nontrivial since $b\vee c>a\geq 0$ and $e>d\geq 0$; hence we are in one of the following subcases:

- $b\vee c\neq 1$ and $e\neq 1$; then $1\notin\{a,b,c,b\vee c,d,e\}$, so $1/Cg_{\mathbb{WCL},L}(a,b)=\{1\}$, thus $(c\vee d)/Cg_{\mathbb{WCL},L}(a,b)=(b\vee c\vee e)/Cg_{\mathbb{WCL},L}(a,b)=1/Cg_{\mathbb{WCL},L}(a,b)=\{1\}$, so $c\vee d=1$, thus $c\| d$ since $c<b\vee c\leq 1\geq e>d$, hence $c\wedge d\notin\{a,b,c,b\vee c,d,e\}$, so $(c\wedge d)/Cg_{\mathbb{WCL},L}(a,b)=\{c\wedge d\}$, thus $((b\vee c)\wedge e)/Cg_{\mathbb{WCL},L}(a,b)=(c\wedge d)/Cg_{\mathbb{WCL},L}(a,b)=\{c\wedge d\}$, hence $(b\vee c)\wedge e=c\wedge d$, therefore $\{c\wedge d,c,d,e,b\vee c,1\}$ is a sublattice of L isomorphic to the hexagon: $\mathcal{C}_4\boxplus\mathcal{C}_4$;
- $b\vee c<1=e>d$, thus $d^\Delta\neq 0$ and $(d^\Delta,0)=(d^\Delta,e^\Delta)\in Cg_{\mathbb{WCL},L}(a,b)=\varepsilon_L(a,b)\cup\varepsilon_L(c,b\vee c)\cup\varepsilon_L(d,e)$, hence $a=0$ and $d^\Delta=b\leq b\vee c$, hence $b^\Delta\leq d\geq(b\vee c)^\Delta$; but $b^\Delta\in a^\Delta/Cg_{\mathbb{WCL},L}(a,b)=1/Cg_{\mathbb{WCL},L}(a,b)=e/Cg_{\mathbb{WCL},L}(a,b)=\{d,e\}$, hence $b^\Delta=d$, while $b\vee c<1$ and $(b\vee c)^\Delta=d<1$ imply $b\vee c\|d$, so $b\vee c>(b\vee c)\wedge d\in((b\vee c)\wedge 1)/Cg_{\mathbb{WCL},L}(a,b)=(b\vee c)/Cg_{\mathbb{WCL},L}(a,b)=\{c,b\vee c\}$, thus $(b\vee c)\wedge d=c$, hence $\{0,b,c,b\vee c,d,1\}=\{a,b,c,b\vee c,d,e\}\cong\mathcal{C}_2\times\mathcal{C}_3$;
- $c<b\vee c=1>e$, thus $c^\Delta\neq 0$ and $(c^\Delta,0)=(c^\Delta,(b\vee c)^\Delta)\in Cg_{\mathbb{WCL},L}(a,b)=\varepsilon_L(a,b)\cup\varepsilon_L(c,b\vee c)\cup\varepsilon_L(d,e)$; $\mathrm{Min}(\{a,b,c,b\vee c,d,e\})\subseteq\{a,d\}$, but we can't have $a=0$, because then we would get the contradiction $d,e\notin L=[0,1]_L=[a,b\vee c]_L=\{a,b,c,b\vee c\}$, thus $d=0$, hence $c^\Delta=e$; thus $e^\Delta\leq c$ and $e^\Delta\in a^\Delta/Cg_{\mathbb{WCL},L}(a,b)=0^\Delta/Cg_{\mathbb{WCL},L}(a,b)=1/Cg_{\mathbb{WCL},L}(a,b)=\{c,b\vee c\}$, thus $e^\Delta=c$; also, $e\nleq a$ since $a<c$ and $c\vee e=c\vee c^\Delta=1$, and $a\vee e\in(a\vee 0)/Cg_{\mathbb{WCL},L}(a,b)=a/Cg_{\mathbb{WCL},L}(a,b)=\{a,b\}$, thus $a\vee e=b$, hence $\{0,e,a,b,c,1\}=\{d,e,a,b,c,b\vee c\}\cong\mathcal{C}_2\times\mathcal{C}_3$.

Theorem 7.1 [3, 5, 19] *If* $n\in\mathbb{N}^*$ *and* L *is a lattice with* $|L|=n$, *then*:

- $|\mathrm{Con}(L)|\leq 2^{n-1}$;
- $|\mathrm{Con}(L)|=2^{n-1}$ *iff* $\mathrm{Con}(L)\cong\mathcal{C}_2^{n-1}$ *iff* $L\cong\mathcal{C}_n$;
- *if* $|\mathrm{Con}(L)|<2^{n-1}$, *then* $|\mathrm{Con}(L)|\leq 2^{n-2}$;
- $|\mathrm{Con}(L)|=2^{n-2}$ *iff* $n\geq 4$ *and* $\mathrm{Con}(L)\cong\mathcal{C}_2^{n-2}$ *iff* $n\geq 4$ *and* $L\cong\mathcal{C}_{n-k-2}\oplus\mathcal{C}_2^2\oplus\mathcal{C}_k$ *for some* $k\in[1,n-3]$;
- *if* $|\mathrm{Con}(L)|<2^{n-2}$, *then* $|\mathrm{Con}(L)|\leq 5\cdot 2^{n-5}$;
- $|\mathrm{Con}(L)|=5\cdot 2^{n-5}$ *iff* $n\geq 5$ *and* $\mathrm{Con}(L)\cong\mathcal{C}_2^{n-5}\times(\mathcal{C}_2\oplus\mathcal{C}_2^2)$ *iff* $n\geq 5$ *and* $L\cong\mathcal{C}_{n-k-3}\oplus\mathcal{N}_5\oplus\mathcal{C}_k$ *for some* $k\in[1,n-4]$;
- *if* $|\mathrm{Con}(L)|<5\cdot 2^{n-5}$, *then* $|\mathrm{Con}(L)|\leq 2^{n-3}$;
- $|\mathrm{Con}(L)|=2^{n-3}$ *iff* $n\geq 6$ *and* $\mathrm{Con}(L)\cong\mathcal{C}_2^{n-3}$ *iff* $n\geq 6$ *and* $L\cong\mathcal{C}_{n-k-4}\oplus(\mathcal{C}_2\times\mathcal{C}_3)\oplus\mathcal{C}_k$ *for some* $k\in[1,n-5]$ *or* $n\geq 7$ *and* $L\cong\mathcal{C}_{n-r-s-4}\oplus\mathcal{C}_2^2\oplus\mathcal{C}_r\oplus\mathcal{C}_2^2\oplus\mathcal{C}_s$ *for some* $r,s\in\mathbb{N}^*$ *with* $r+s\leq n-5$;
- *if* $|\mathrm{Con}(L)|<2^{n-3}$, *then* $|\mathrm{Con}(L)|\leq 7\cdot 2^{n-6}$;
- $|\mathrm{Con}(L)|=7\cdot 2^{n-6}$ *iff* $n\geq 6$ *and* $\mathrm{Con}(L)\cong\mathcal{C}_2^{n-6}\times(\mathcal{C}_2^2\oplus\mathcal{C}_2^2)$ *iff* $n\geq 6$ *and, for some* $k\in[1,n-5]$, *either* $L\cong\mathcal{C}_{n-k-4}\oplus(\mathcal{C}_3\boxplus\mathcal{C}_5)\oplus\mathcal{C}_k$ *or* $L\cong\mathcal{C}_{n-k-4}\oplus(\mathcal{C}_4\boxplus\mathcal{C}_4)\oplus\mathcal{C}_k$.

Theorem 7.2 *For any $n \in \mathbb{N}^*$, any lattice L with $|L| = n$ and any weak complementation $^{\Delta}$ on L, we have:*

1. $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| \leq 2^{n-2} + 1$;
2. $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| = 2^{n-2} + 1$ *iff* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^{n-2} \oplus \mathcal{C}_2$ *iff* $n \geq 2$ *and* $L \cong \mathcal{C}_n$;
3. $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| = 2^{n-2}$ *iff* $n = 4$ *and* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^2$ *iff* $L \cong \mathcal{C}_2^2$ *and* $^{\Delta} = ^{\Delta L \setminus \{1\}}$ *is the Boolean complementation*;
4. *if* $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| < 2^{n-2}$, *then* $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| \leq 2^{n-3} + 1$;
5. $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| = 2^{n-3} + 1$ *iff* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^{n-3} \oplus \mathcal{C}_2$ *iff* $n \geq 5$ *and* $L \cong \mathcal{C}_{n-k-2} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ *for some* $k \in [2, n-3]$;
6. $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| = 3 \cdot 2^{n-5}$ *iff* $n = 5$ *and* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_3$ *or* $n = 6$ *and* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2 \times \mathcal{C}_3$ *iff* $L \cong \mathcal{N}_5$ *or* $L \cong \mathcal{C}_2 \times \mathcal{C}_3$ *and* $^{\Delta} = ^{\Delta \mathcal{C}_2 \times \Delta \mathcal{C}_3}$ *is the direct product of the trivial weak complementations on the chains* $\mathcal{C}_2$ *and* $\mathcal{C}_3$;
7. *if* $L \ncong \mathcal{N}_5$, $(L,^{\Delta}) \ncong_{\mathbb{WCL}} (\mathcal{C}_2,^{\Delta \mathcal{C}_2}) \times (\mathcal{C}_3,^{\Delta \mathcal{C}_3})$ *and* $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| \leq 2^{n-3}$, *then* $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| \leq 5 \cdot 2^{n-6} + 1$;
8. $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| = 5 \cdot 2^{n-6} + 1$ *iff* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong (\mathcal{C}_2^{n-6} \times (\mathcal{C}_2 \oplus \mathcal{C}_2^2)) \oplus \mathcal{C}_2$ *iff* $n \geq 6$ *and* $L \cong \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k$ *for some* $k \in [2, n-4]$;
9. $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| = 5 \cdot 2^{n-6}$ *iff* $n = 6$ *and either* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2 \oplus \mathcal{C}_2^2$ *or* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^2 \oplus \mathcal{C}_2$ *iff one of the following holds*:
   - $L \cong \mathcal{C}_2 \times \mathcal{C}_3$ *and* $^{\Delta} = ^{\Delta L \setminus \{1\}}$, *case in which* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2 \oplus \mathcal{C}_2^2$;
   - $L \cong \mathcal{C}_3 \boxplus \mathcal{C}_5$ *or* $L \cong \mathcal{C}_4 \boxplus \mathcal{C}_4$ *and* $^{\Delta} = ^{\Delta L}$ *is trivial, case in which* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^2 \oplus \mathcal{C}_2$;
10. *if* $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| < 5 \cdot 2^{n-6}$, *then* $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| \leq 2^{n-4} + 1$;
11. $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| = 2^{n-4} + 1$ *iff* $n \geq 5$ *and* $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^{n-4} \oplus \mathcal{C}_2$ *iff one of the following holds*:
    - $n \geq 5$, $L \cong \mathcal{C}_{n-r-s+3} \oplus (\mathcal{C}_r \boxplus \mathcal{C}_s)$ *for some* $r, s \in \mathbb{N} \setminus \{0, 1, 2\}$ *such that* $r + s \leq n + 2$ *and, if* $r + s > 6$ *(that is if* $L \ncong \mathcal{C}_{n-3} \oplus \mathcal{C}_2^2$*), then* $^{\Delta} = ^{\Delta L}$ *is trivial;*
    - $n \geq 7$ *and* $L \cong \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_k$ *for some* $k \in [2, n-5]$*;*
    - $n \geq 8$ *and* $L \cong \mathcal{C}_{n-r-s-4} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_r \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_s$ *for some* $r, s \in \mathbb{N}^*$ *such that* $s > 1$ *and* $r + s \leq n - 5$*.*

Dually for congruences of dual weakly complemented lattices.

Proof For any bounded lattice L, any weak complementation $^{\Delta}$ on L and any $\alpha \in \mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})$, we will denote by $(L/\alpha,^{\Delta[\alpha]}) = (L,^{\Delta})/\alpha$. Of course, the trivial weak complementation of the quotient L/α is $^{\Delta L/\alpha} = ^{\Delta L[\alpha]}$.

For every $n \in \mathbb{N}^*$, $\mathcal{C}_n$ can only be endowed with the trivial weak complementation $^{\Delta \mathcal{C}_n}$ and $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_n,^{\Delta \mathcal{C}_n}) = \mathrm{Con}_1(\mathcal{C}_n) \cup \{\mathcal{C}_n^2\}$, thus $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_1,^{\Delta \mathcal{C}_1}) \cong \mathcal{C}_1$ and, if $n \geq 2$, then $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_n,^{\Delta \mathcal{C}_n}) \cong \mathrm{Con}(\mathcal{C}_{n-1}) \oplus \mathcal{C}_2 \cong \mathcal{C}_2^{n-2} \oplus \mathcal{C}_2$, hence $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_1,^{\Delta \mathcal{C}_1})| = 1 < 2^{-1} + 1$ and, if $n \geq 2$, then $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_n,^{\Delta \mathcal{C}_n})| = 2^{n-2} + 1$.

Also, $\mathcal{C}_2^2 \oplus \mathcal{C}_2$ has the 1 join–irreducible, thus it can only be endowed with the trivial weak complementation, and $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2^2 \oplus \mathcal{C}_2,^{\Delta \mathcal{C}_2^2 \oplus \mathcal{C}_2}) = \mathrm{Con}_1(\mathcal{C}_2^2 \oplus \mathcal{C}_2) \cup \{(\mathcal{C}_2^2 \oplus \mathcal{C}_2)^2\} \cong \mathcal{C}_2^2 \oplus \mathcal{C}_2$, so $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2^2 \oplus \mathcal{C}_2,^{\Delta \mathcal{C}_2^2 \oplus \mathcal{C}_2})| = 2^2 + 1$.

By the above, $\mathcal{M}_3 \cong \mathcal{C}_3 \boxplus \mathcal{C}_3 \boxplus \mathcal{C}_3 \cong \mathcal{C}_3 \boxplus \mathcal{C}_2^2$ can only be endowed with $^{\Delta \mathcal{M}_3}$, and $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{M}_3,^{\Delta \mathcal{M}_3}) = \mathrm{Con}_1(\mathcal{M}_3) \cup \{\mathcal{M}_3^2\} \cong \mathcal{C}_1 \oplus \mathcal{C}_2 \cong \mathcal{C}_2$, so $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{M}_3,^{\Delta \mathcal{M}_3})| = 2^1$.

In what follows, as above, we will use the remarks in Section 4 and Proposition 6.1 to determine the weak complementations on the following lattices and Lemma 4.1 and

Proposition 6.2 to determine the congruences of the weakly complemented lattices formed with those weak complementations. The cases not covered by these results need to be verified directly.

$\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2^2,{}^{\Delta\mathcal{C}_2^2}) = \mathrm{Con}_1(\mathcal{C}_2^2) \cup \{(\mathcal{C}_2^2)^2\} \cong \mathcal{C}_1 \oplus \mathcal{C}_2 \cong \mathcal{C}_2$, so $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2^2,{}^{\Delta\mathcal{C}_2^2})| = 2^1 = 2^0 + 1$, while $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2^2,{}^{\Delta\mathcal{C}_2^2\setminus\{1\}}) \cong \mathcal{C}_2^2$ since $(\mathcal{C}_2^2,{}^{\Delta\mathcal{C}_2^2\setminus\{1\}})$ is a Boolean algebra, so $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2^2,{}^{\Delta\mathcal{C}_2^2\setminus\{1\}})| = 2^2$.

$\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2 \oplus \mathcal{C}_2^2,{}^{\Delta\mathcal{C}_2\oplus\mathcal{C}_2^2}) = \mathrm{Con}_1(\mathcal{C}_2 \oplus \mathcal{C}_2^2) \cup \{(\mathcal{C}_2 \oplus \mathcal{C}_2^2)^2\} \cong \mathcal{C}_2 \oplus \mathcal{C}_2 \cong \mathcal{C}_3$, so $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2 \oplus \mathcal{C}_2^2,{}^{\Delta\mathcal{C}_2\oplus\mathcal{C}_2^2})| = 2^1 + 1$, while $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2 \oplus \mathcal{C}_2^2,{}^{\Delta(\mathcal{C}_2\oplus\mathcal{C}_2^2)\setminus\{1\}}) \cong (\mathrm{Con}(\mathcal{C}_2) \times \mathrm{Con}_{\mathbb{WCL}1}(\mathcal{C}_2^2,{}^{\Delta\mathcal{C}_2^2\setminus\{1\}})) \oplus \mathcal{C}_2 \cong (\mathcal{C}_2 \times \mathcal{C}_1) \oplus \mathcal{C}_2 \cong \mathcal{C}_2 \oplus \mathcal{C}_2 \cong \mathcal{C}_3$, so $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_2 \oplus \mathcal{C}_2^2,{}^{\Delta(\mathcal{C}_2\oplus\mathcal{C}_2^2)\setminus\{1\}})| = 2^1 + 1$.

Recall from Example 6.1 that $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{N}_5,{}^{\Delta\mathcal{N}_5}) = \mathrm{Con}_{\mathbb{WCL}}(\mathcal{N}_5,{}^{\Delta\mathcal{N}_5\setminus\{1\}}) \cong \mathcal{C}_3$, thus $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{N}_5,{}^{\Delta\mathcal{N}_5})| = |\mathrm{Con}_{\mathbb{WCL}}(\mathcal{N}_5,{}^{\Delta\mathcal{N}_5\setminus\{1\}})| = 3$, and note that $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_3 \boxplus \mathcal{C}_5,{}^{\Delta\mathcal{C}_3\boxplus\mathcal{C}_5}) = \mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_3 \boxplus \mathcal{C}_5,{}^{\Delta(\mathcal{C}_3\boxplus\mathcal{C}_5)\setminus\{1\}}) \cong \mathcal{C}_2^2 \oplus \mathcal{C}_2 \cong \mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_4 \boxplus \mathcal{C}_4,{}^{\Delta\mathcal{C}_4\boxplus\mathcal{C}_4}) = \mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_4 \boxplus \mathcal{C}_4,{}^{\Delta(\mathcal{C}_4\boxplus\mathcal{C}_4)\setminus\{1\}})$, thus $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_3 \boxplus \mathcal{C}_5,{}^{\Delta\mathcal{C}_3\boxplus\mathcal{C}_5})| = |\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_3 \boxplus \mathcal{C}_5,{}^{\Delta(\mathcal{C}_3\boxplus\mathcal{C}_5)\setminus\{1\}})| = |\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_4 \boxplus \mathcal{C}_4,{}^{\Delta\mathcal{C}_4\boxplus\mathcal{C}_4})| = |\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_4 \boxplus \mathcal{C}_4,{}^{\Delta(\mathcal{C}_4\boxplus\mathcal{C}_4)\setminus\{1\}})| = 5$.

See the weak complementations on $\mathcal{C}_2 \times \mathcal{C}_3$ and their corresponding congruence lattices in Example 5.3.

By Theorem 7.1 and some more quick verifications:

• the six–element lattices with at least $2^2 + 1 = 5$ lattice congruences are $\mathcal{C}_6$, $\mathcal{C}_2 \times \mathcal{C}_3$, $\mathcal{C}_2^2 \oplus \mathcal{C}_3$, $\mathcal{C}_2 \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_2$, $\mathcal{C}_3 \oplus \mathcal{C}_2^2$, $\mathcal{N}_5 \oplus \mathcal{C}_2$ and $\mathcal{C}_2 \oplus \mathcal{N}_5$, $\mathcal{C}_3 \boxplus \mathcal{C}_5$ and $\mathcal{C}_4 \boxplus \mathcal{C}_4$;

• the seven–element lattices with at least $2^3 + 1 = 9$ lattice congruences are $\mathcal{C}_7$, $(\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_2$, $\mathcal{C}_2 \oplus (\mathcal{C}_2 \times \mathcal{C}_3)$, $\mathcal{C}_2^2 \oplus \mathcal{C}_4$, $\mathcal{C}_2 \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_3$, $\mathcal{C}_3 \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_2$, $\mathcal{C}_4 \oplus \mathcal{C}_2^2$, $\mathcal{C}_2^2 \oplus \mathcal{C}_2^2$, $\mathcal{N}_5 \oplus \mathcal{C}_3$, $\mathcal{C}_2 \oplus \mathcal{N}_5 \oplus \mathcal{C}_2$, $\mathcal{C}_3 \oplus \mathcal{N}_5$, $(\mathcal{C}_3 \boxplus \mathcal{C}_5) \oplus \mathcal{C}_2$, $\mathcal{C}_2 \oplus (\mathcal{C}_3 \boxplus \mathcal{C}_5)$, $(\mathcal{C}_4 \boxplus \mathcal{C}_4) \oplus \mathcal{C}_2$, $\mathcal{C}_2 \oplus (\mathcal{C}_4 \boxplus \mathcal{C}_4)$, $\mathcal{C}_3 \boxplus \mathcal{C}_6$ and $\mathcal{C}_4 \boxplus \mathcal{C}_5$;

• the eight–element lattices with at least $2^4 + 1 = 17$ lattice congruences are $\mathcal{C}_8$, $\mathcal{C}_2 \times \mathcal{C}_4$, $(\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_3$, $\mathcal{C}_2 \oplus (\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_2$, $\mathcal{C}_3 \oplus (\mathcal{C}_2 \times \mathcal{C}_3)$, $\mathcal{C}_2^2 \oplus \mathcal{C}_5$, $\mathcal{C}_2 \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_4$, $\mathcal{C}_3 \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_3$, $\mathcal{C}_4 \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_2$, $\mathcal{C}_5 \oplus \mathcal{C}_2^2$, $\mathcal{C}_2^2 \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_2$, $\mathcal{C}_2^2 \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^2$, $\mathcal{C}_2 \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_2^2$, $\mathcal{N}_5 \oplus \mathcal{C}_4$, $\mathcal{C}_2 \oplus \mathcal{N}_5 \oplus \mathcal{C}_3$, $\mathcal{C}_3 \oplus \mathcal{N}_5 \oplus \mathcal{C}_2$, $\mathcal{C}_4 \oplus \mathcal{N}_5$, $\mathcal{C}_2^2 \oplus \mathcal{N}_5$, $\mathcal{N}_5 \oplus \mathcal{C}_2^2$, $(\mathcal{C}_3 \boxplus \mathcal{C}_5) \oplus \mathcal{C}_3$, $\mathcal{C}_2 \oplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \oplus \mathcal{C}_2$, $\mathcal{C}_3 \oplus (\mathcal{C}_3 \boxplus \mathcal{C}_5)$, $(\mathcal{C}_4 \boxplus \mathcal{C}_4) \oplus \mathcal{C}_3$, $\mathcal{C}_2 \oplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \oplus \mathcal{C}_2$, $\mathcal{C}_3 \oplus (\mathcal{C}_4 \boxplus \mathcal{C}_4)$, $(\mathcal{C}_3 \boxplus \mathcal{C}_6) \oplus \mathcal{C}_2$, $\mathcal{C}_2 \oplus (\mathcal{C}_3 \boxplus \mathcal{C}_6)$, $(\mathcal{C}_4 \boxplus \mathcal{C}_5) \oplus \mathcal{C}_2$, $\mathcal{C}_2 \oplus (\mathcal{C}_4 \boxplus \mathcal{C}_5)$, $\mathcal{C}_3 \boxplus \mathcal{C}_7$, $\mathcal{C}_4 \boxplus \mathcal{C}_6$ and $\mathcal{C}_5 \boxplus \mathcal{C}_5$;

and, with calculations similar to the above, one can easily check that all these lattices satisfy the statements in the enunciation. Note that, out of all the weakly complemented lattices $(L,{}^{\Delta})$ above, the only one that has exactly $2^{|L|-2}$ congruences is $(\mathcal{C}_2^2,{}^{\Delta\mathcal{C}_2^2\setminus\{1\}})$, the only ones that have exactly $3 \cdot 2^{|L|-5}$ congruences are $\mathcal{N}_5$ endowed with any of its weak complementations and $\mathcal{C}_2 \times \mathcal{C}_3$ endowed with the direct product ${}^{\Delta\mathcal{C}_2\times\Delta\mathcal{C}_3}$ and the only ones that have exactly $5 \cdot 2^{|L|-6}$ congruences are $\mathcal{C}_2 \times \mathcal{C}_3$ endowed with the weak complementation ${}^{\Delta(\mathcal{C}_2\times\mathcal{C}_3)\setminus\{1\}}$ (which has the congruence lattice isomorphic to $\mathcal{C}_2 \oplus \mathcal{C}_2^2$ by Example 5.3) and $\mathcal{C}_3 \boxplus \mathcal{C}_5$ and $\mathcal{C}_4 \boxplus \mathcal{C}_4$ endowed with their trivial weak complementations (having the congruence lattices isomorphic to $\mathcal{C}_2^2 \oplus \mathcal{C}_2$).

Now let $n \in \mathbb{N}$ such that $n \geq 9$ and assume that the statements in the enunciation hold for any lattice whose cardinality is at most $n - 1$.

Let $(L,{}^{\Delta})$ be a weakly complemented lattice with $|L| = n$ and $\alpha \in \mathrm{At}(\mathrm{Con}_{\mathbb{WCL}}(L,{}^{\Delta})) \subseteq \mathrm{Con}_{\mathbb{WCL}}(L,{}^{\Delta}) \setminus \{=_L\}$, so that $|L/\alpha| \leq n - 1$ and, according to Remark 7.1, $\alpha = Cg_{\mathbb{WCL},L}(a, b)$ for some $a, b \in L$ with $a \prec b$.

(i) By the induction hypothesis, $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| \leq 2^{n-3} + 1$ and thus $|\mathrm{Con}_{\mathbb{WCL}}(L,{}^{\Delta})| \leq 2^{n-2} + 2$ according to Lemma 7.1.

Assume by absurdum that $|\mathrm{Con}_{\mathbb{WCL}}(L)| = 2^{n-2} + 2$. Then, by Lemma 7.1, it follows that $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| \geq 2^{n-3} + 1$, hence $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| = 2^{n-3} + 1$ and thus $L/\alpha \cong \mathcal{C}_{n-1}$

by the induction hypothesis, in particular $|L/\alpha| = n-1 = |L|-1$, thus, by Lemma 7.3, (i), $a \in \mathrm{Mi}(L)$, $b \in \mathrm{Ji}(L)$, $a^\Delta = b^\Delta$ and $\alpha = Cg_L(a,b) = \varepsilon_L(a,b)$.

Since L/α is a chain, it follows that, for any $x, y \in L \setminus \{a,b\}$, $x/\alpha = \{x\}$ is comparable to $y/\alpha = \{y\}$ and to $a/\alpha = b/\alpha = \{a,b\}$, hence x is comparable to y and to at least one of a and b. Assume by absurdum that $x \| a$ or $x \| b$. W.l.g. we may assume that $x \| a$. Then, by the above, $x < b$, $x > a \wedge x < a$ and $a < a \vee x \leq b$, thus $a \vee x = b$, hence $\{a \wedge x, a, x, b\}$ is a sublattice of L isomorphic to $\mathcal{C}_2^2$, therefore $(a \wedge x, x) \in \alpha$, which contradicts the fact that α only collapses a with b. Hence x is comparable to each of a and b.

Therefore $L \cong \mathcal{C}_n$, thus ${}^\Delta = {}^{\Delta L}$ and $\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta) = \mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L}) \cong \mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_n, {}^{\Delta \mathcal{C}_n}) \cong \mathcal{C}_2^{n-2} \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L})| = 2^{n-2}+1$, contradicting the above.

Hence $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| \leq 2^{n-2}+1$.

In what follows, one can reason as above on the lattice structure of L based on that of L/α and the form of α; we will skip such details in the rest of this proof.

(ii) Assume that $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| = 2^{n-2}+1$. By Theorem 7.1, if L were not a chain, then $|\mathrm{Con}(L)| \leq 2^{n-2} < |\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)|$, which would contradict the fact that $\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta) \subseteq \mathrm{Con}(L)$.

Hence, by the proof of (i), $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| = 2^{n-2}+1$ exactly when $L \cong \mathcal{C}_n$.

(iii) Assume by absurdum that $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| = 2^{n-2} < 2^{n-2}+1$, so that $L \ncong \mathcal{C}_n$ by (ii), thus $|\mathrm{Con}(L)| \leq 2^{n-2}$ by Theorem 7.1, hence $\mathrm{Con}(L) = \mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)$, in particular $|\mathrm{Con}(L)| = |\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| = 2^{n-2}$, therefore, again by Theorem 7.1, $L \cong \mathcal{C}_{n-k-2} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ for some $k \in [1, n-3]$.

If ${}^\Delta = {}^{\Delta L}$, then $\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta) = \mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L}) = \mathrm{Con}_1(L) \cup \{L^2\} \neq \mathrm{Con}(L)$, which contradicts the above. Hence ${}^\Delta \neq {}^{\Delta L}$, thus $k = 1$, so $L \cong \mathcal{C}_{n-3} \oplus \mathcal{C}_2^2$, and ${}^\Delta = {}^{\Delta(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2) \setminus \{1\}}$. Then, since $n \geq 9 > 4$ and thus $\mathcal{C}_{n-3}$ is nonsingleton, $\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta) = \mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2) \setminus \{1\}}) \cong (\mathrm{Con}(\mathcal{C}_{n-3}) \times \mathrm{Con}_{\mathbb{WCL}1}(\mathcal{C}_2^2, {}^{\Delta \mathcal{C}_2^2 \setminus \{1\}})) \oplus \mathcal{C}_2 \cong (\mathcal{C}_2^{n-4} \times \mathcal{C}_1) \oplus \mathcal{C}_2 \cong \mathcal{C}_2^{n-4} \oplus \mathcal{C}_2$, so $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| = 2^{n-4}+1 \neq 2^{n-2}$ and we have another contradiction.

Therefore $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| \neq 2^{n-2}$.

(iv) Assume that $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| < 2^{n-2}+1$, so that $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| < 2^{n-2}$ by (iii) and $L \ncong \mathcal{C}_n$ by (ii).

Assume by absurdum that $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| \geq 2^{n-3}+2$, so that, by Lemma 7.1, $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha, {}^{\Delta[\alpha]})| \geq 2^{n-4}+1 > 4 = 2^{4-2}$ since $n \geq 9$, thus, by the induction hypothesis, we are in one of the following cases:

- $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha, {}^{\Delta[\alpha]})| = 2^{n-3}+1$, but then, as in (i), we would obtain that $L \cong \mathcal{C}_n$, contradicting the above;
- $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha, {}^{\Delta[\alpha]})| = 2^{n-4}+1$, hence we are in one of the following subcases:
  - $|L/\alpha| = n-1$, so that α is as in Lemma 7.3, (i), and $L/\alpha \cong \mathcal{C}_{n-k-3} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ for some $k \in [2, n-4]$, hence $L \cong \mathcal{C}_{n-k-2} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ or $L \cong \mathcal{C}_{n-k-3} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_{k+1}$ or $L \cong \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k$, so ${}^{\Delta L}$ is the only weak complementation on L and, in the first two of these cases, $\mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L}) \cong \mathcal{C}_2^{n-3} \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L})| = 2^{n-3}+1$, while, in the latter case, $\mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L}) \cong \mathrm{Con}(\mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_{k-1}) \oplus \mathcal{C}_2 \cong (\mathcal{C}_2^{n-6} \times (\mathcal{C}_2 \oplus \mathcal{C}_2^2)) \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L})| = 5 \cdot 2^{n-6}+1$;
  - $|L/\alpha| = n-2 > 4$ and $L/\alpha \cong \mathcal{C}_{n-2}$, so that α is as in Lemma 7.3, (ii), case (β_1), thus $L \cong \mathcal{C}_{n-k-2} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ for some $k \in [1, n-3]$ and ${}^\Delta = {}^{\Delta L}$, thus, as above, $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L})| = 2^{n-3}+1$ if $k \geq 2$, while, if $k = 1$, so that $L \cong \mathcal{C}_{n-3} \oplus \mathcal{C}_2^2$, then $\mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L}) \cong (\mathcal{C}_2^{n-4} \times \mathcal{C}_1) \oplus \mathcal{C}_2 \cong \mathcal{C}_2^{n-4} \oplus \mathcal{C}_2$, so that $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^{\Delta L})| = 2^{n-4}+1$.

Every case above contradicts the assumption that $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| = 2^{n-3}+2$; hence $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| \leq 2^{n-3}+1$.

(v) Assume that $|\mathrm{Con}_{\mathbb{WCL}}(L, {}^\Delta)| = 2^{n-3}+1$, so that $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha, {}^{\Delta[\alpha]})| \geq 2^{n-4}+1/2$ by Lemma 7.1, thus $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha, {}^{\Delta[\alpha]})| \geq 2^{n-4}+1$ and hence, by the proof of (iv),

$L \cong \mathcal{C}_{n-k-2} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ for some $k \in [1, n-3]$ or $L \cong \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k$ for some $k \in [2, n-4]$, out of which only the first form for $k \geq 2$, with its unique weak complementation ${}^{\Delta L}$, has exactly $2^{n-3}+1$ congruences.

(vi),(vii) Assume that $|\mathrm{Con}_{\mathbb{WCL}}(L,{}^{\Delta})| \leq 2^{n-3}$, so that, by (i)–(v), L is not isomorphic to either of the lattices $\mathcal{C}_n$ and $\mathcal{C}_{n-k-2} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ for any $k \in [2, n-4]$.

Assume by absurdum that $|\mathrm{Con}_{\mathbb{WCL}}(L,{}^{\Delta})| \geq 5 \cdot 2^{n-6}+2$, so that $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| \geq 5 \cdot 2^{n-7}+1$ by Lemma 7.1, so that, by the induction hypothesis, we are in one of the following cases:

■ $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| = 2^{n-3}+1$; but then, as in (i), it would follow that $L \cong \mathcal{C}_n$, contradicting the above;

■ $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| = 2^{n-4}+1$; but then, as in (iv), it would follow that $L \cong \mathcal{C}_{n-k-2} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ for some $k \in [1, n-3]$ or $L \cong \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k$ for some $k \in [2, n-4]$, hence, by the cases eliminated at the beginning of this proof of (vi),(vii), either $L \cong \mathcal{C}_{n-3} \oplus \mathcal{C}_2^2$, which can be endowed with ${}^{\Delta L}$ and with ${}^{\Delta L \setminus \{1\}}$, or $L \cong \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k$ for some $k \in [2, n-4]$, which can only be endowed with ${}^{\Delta L}$, and neither of these weakly complemented lattices has $5 \cdot 2^{n-6}+2$ or more congruences; indeed:

$\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2,{}^{\Delta \mathcal{C}_{n-3} \oplus \mathcal{C}_2^2}) \cong \mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2,{}^{\Delta(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2)\setminus\{1\}}) \cong (\mathrm{Con}(\mathcal{C}_{n-3}) \times \mathcal{C}_1) \oplus \mathcal{C}_2 \cong \mathcal{C}_2^{n-4} \oplus \mathcal{C}_2$, hence $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2,{}^{\Delta \mathcal{C}_{n-3} \oplus \mathcal{C}_2^2})| = |\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2,{}^{\Delta(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2)\setminus\{1\}})| = 2^{n-4}+1$;

$\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k,{}^{\Delta \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k}) \cong \mathrm{Con}(\mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_{k-1}) \oplus \mathcal{C}_2 \cong (\mathcal{C}_2^{n-6} \times (\mathcal{C}_2 \oplus \mathcal{C}_2^2)) \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k,{}^{\Delta \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k})| = 5 \cdot 2^{n-6}+1$;

■ $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| = 5 \cdot 2^{n-7}+1$; then, by the induction hypothesis, it follows that $|L/\alpha| = n-1$ and $L/\alpha \cong \mathcal{C}_{n-k-4} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k$ for some $k \in [2, n-5]$, so that, by Lemma 7.3, (i), $L \cong \mathcal{C}_{n-k-4} \oplus \mathcal{N}_5 \oplus \mathcal{C}_{k+1}$ or $L \cong \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k$ or $L \cong \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \oplus \mathcal{C}_k$ or $L \cong \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \oplus \mathcal{C}_k$, each of which can only be endowed with the trivial weak complementation ${}^{\Delta L}$, w.r.t. which neither has $5 \cdot 2^{n-6}+2$ or more congruences; indeed:

$\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-k-4} \oplus \mathcal{N}_5 \oplus \mathcal{C}_{k+1},{}^{\Delta \mathcal{C}_{n-k-4} \oplus \mathcal{N}_5 \oplus \mathcal{C}_{k+1}}) \cong \mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k,{}^{\Delta \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k}) \cong (\mathcal{C}_2^{n-6} \times (\mathcal{C}_2 \oplus \mathcal{C}_2^2)) \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-k-4} \oplus \mathcal{N}_5 \oplus \mathcal{C}_{k+1},{}^{\Delta \mathcal{C}_{n-k-4} \oplus \mathcal{N}_5 \oplus \mathcal{C}_{k+1}})| = |\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k,{}^{\Delta \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k})| = 5 \cdot 2^{n-6}+1$;

it is easy to check that $\mathrm{Con}(\mathcal{C}_3 \boxplus \mathcal{C}_5) \cong \mathrm{Con}(\mathcal{C}_4 \boxplus \mathcal{C}_4) \cong \mathcal{C}_2^2 \oplus \mathcal{C}_2^2$ [19], so that: $\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-k-4} \oplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \oplus \mathcal{C}_k,{}^{\Delta \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \oplus \mathcal{C}_k}) \cong \mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-k-4} \oplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \oplus \mathcal{C}_k,{}^{\Delta \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \oplus \mathcal{C}_k}) \cong (\mathcal{C}_2^{n-7} \times (\mathcal{C}_2^2 \oplus \mathcal{C}_2^2)) \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-k-4} \oplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \oplus \mathcal{C}_k,{}^{\Delta \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \oplus \mathcal{C}_k})| = |\mathrm{Con}_{\mathbb{WCL}}(\mathcal{C}_{n-k-4} \oplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \oplus \mathcal{C}_k,{}^{\Delta \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \oplus \mathcal{C}_k})| = 7 \cdot 2^{n-7}+1$.

Therefore $|\mathrm{Con}_{\mathbb{WCL}}(L,{}^{\Delta})| \leq 5 \cdot 2^{n-6}+1$.

(viii) Assume that $|\mathrm{Con}_{\mathbb{WCL}}(L,{}^{\Delta})| = 5 \cdot 2^{n-6}+1$, so that $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| \geq 5 \cdot 2^{n-7}+1/2$ by Lemma 7.1, and thus $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| \geq 5 \cdot 2^{n-7}+1$, hence, as in the proof of (vii), it follows that $L \cong \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k$ for some $k \in [2, n-4]$.

(ix),(x) Assume that $|\mathrm{Con}_{\mathbb{WCL}}(L,{}^{\Delta})| < 5 \cdot 2^{n-6}$ and assume by absurdum that $|\mathrm{Con}_{\mathbb{WCL}}(L,{}^{\Delta})| \geq 2^{n-4}+2$, so that $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| \geq 2^{n-5}+1$ by Lemma 7.1.

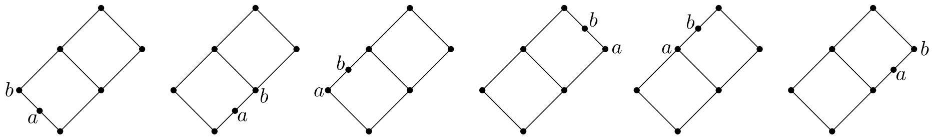
Then, by the proof of (viii), we can not have $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| \in \{2^{n-3}+1, 2^{n-4}+1, 5 \cdot 2^{n-7}+1\}$, hence, by the induction hypothesis, it follows that $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| = 2^{n-5}+1$.

Let us note that, in the case when $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| = 2^{n-4}+1$, one of the possible structures of L is $L \cong \mathcal{C}_{n-3} \oplus \mathcal{C}_2^2$, endowed with any of its two weak complementations, which is the only situation where we have $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| \in \{2^{n-3}+1, 2^{n-4}+1, 5 \cdot 2^{n-7}+1\}$ and $|\mathrm{Con}_{\mathbb{WCL}}(L,{}^{\Delta})| = 2^{n-4}+1$.

Now we consider the case $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,{}^{\Delta[\alpha]})| = 2^{n-5}+1$, in which, by the induction hypothesis, we can be in one of these situations:

■ $|L/\alpha| = n-1$, so that, by the induction hypothesis, Lemma 7.3, (i), the fact that, for any bounded lattice L, if $b = 1 \succ a \in L$, so that $a^{\Delta L} = 1$ and $b^{\Delta L} = 1^{\Delta L} = 0$, then $Cg_{\mathbb{WCL},L}(a,b) = L^2$ in $(L, ^{\Delta L})$ and similar calculations to those in Example 6.1, one of the following holds:

- $L/\alpha \cong \mathcal{C}_{n-r-s+2} \oplus (\mathcal{C}_r \boxplus \mathcal{C}_s)$ for some $r, s \in \mathbb{N} \setminus \{0,1,2\}$ such that $r+s \leq n$, so that $L \cong \mathcal{C}_{n-r-s+3} \oplus (\mathcal{C}_r \boxplus \mathcal{C}_s)$ or $L \cong \mathcal{C}_{n-r-s+2} \oplus (\mathcal{C}_{r+1} \boxplus \mathcal{C}_s)$ or $L \cong \mathcal{C}_{n-r-s+2} \oplus (\mathcal{C}_r \boxplus \mathcal{C}_{s+1})$; each of these three bounded lattices has exactly $2^{n-4}+1$ congruences w.r.t. its trivial weak complementation, and, if it is not isomorphic to $\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2$, then it has strictly less than $2^{n-4}+1$ congruences w.r.t. its nontrivial weak complementation $^{\Delta L \setminus \{1\}}$;
- $L/\alpha \cong \mathcal{C}_{n-k-5} \oplus (\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_k$ for some $k \in [2, n-6]$, so that $L \cong \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_k$ or $L \cong \mathcal{C}_{n-k-5} \oplus (\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_{k+1}$ or $L \cong \mathcal{C}_{n-k-5} \oplus M \oplus \mathcal{C}_k$, where M is one of the following bounded lattices, each of which can be easily proven to have exactly 9 lattice congruences, in whose Hasse diagrams we indicate the pair a, b of elements of L that generates α as in Lemma 7.3, (i):



out of the eight lattices enumerated above, w.r.t. their unique trivial weak complementations, the first two have exactly $2^{n-4}+1$ congruences, while the other six have exactly $9 \cdot 2^{n-8}+1$ congruences;
- $L/\alpha \cong \mathcal{C}_{n-r-s-5} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_r \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_s$ for some $r, s \in \mathbb{N}^*$ such that $s > 1$ and $r+s \leq n-6$, so that $L \cong \mathcal{C}_{n-r-s-4} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_r \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_s$ or $L \cong \mathcal{C}_{n-r-s-5} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_{r+1} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_s$ or $L \cong \mathcal{C}_{n-r-s-5} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_r \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_{s+1}$ or $L \cong \mathcal{C}_{n-r-s-5} \oplus \mathcal{N}_5 \oplus \mathcal{C}_r \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_s$ or $L \cong \mathcal{C}_{n-r-s-5} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_r \oplus \mathcal{N}_5 \oplus \mathcal{C}_s$, each of which can only be endowed with the trivial weak complementation, w.r.t. which the first three have exactly $2^{n-4}+1$ congruences, while the latter two have exactly $5 \cdot 2^{n-7}+1$ congruences;

■ $|L/\alpha| = n-2$, so that, by the induction hypothesis, $L/\alpha \cong \mathcal{C}_{n-k-4} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ for some $k \in [2, n-5]$ and, since $n \geq 9$, we can be in neither of the cases $\textcircled{\alpha}$ or $\textcircled{\beta_2}$ in Lemma 7.3, (ii), thus we are in case $\textcircled{\beta_1}$, hence $L \cong \mathcal{C}_{n-k-h-4} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_h \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ for some $h \in [1, n-k-5]$ or $L \cong \mathcal{C}_{n-k-4} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_h \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_{k-h}$ for some $h \in [1, k-3]$ or $L \cong \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_k$, each of which can only be endowed with the trivial weak complementation, w.r.t. which it has exactly $2^{n-4}+1$ congruences;

■ $|L/\alpha| = n-3$, so that $L/\alpha \cong \mathcal{C}_{n-3}$ by the induction hypothesis and, since $n \geq 9$, we can be in neither of the cases $\textcircled{\gamma}$, $\textcircled{\delta_1}$, $\textcircled{\delta_2}$, $\textcircled{\epsilon_2}$, $\textcircled{\varphi_2}$ or $\textcircled{\psi_2}$ in Lemma 7.3, (iii), hence we are in one of the following cases:

$\textcircled{\delta_3}$, $\textcircled{\delta_4}$, $\textcircled{\epsilon_1}$: in each of these cases $L \cong \mathcal{C}_{n-k-2} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ for some $k \in [1, n-3]$, which contradicts the fact that, with the notations in Lemma 7.3, (iii), $a/Cg_L(a,b) = [a, b \vee c]_L \cong \mathcal{C}_2^2$;

$\textcircled{\varphi_1}$ or $\textcircled{\psi_1}$: $L \cong \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k$ for some $k \in [2, n-4]$, which has $5 \cdot 2^{n-6}+1$ congruences w.r.t. its unique weak complementation, contradicting the hypothesis;

$\textcircled{\chi}$: since L/α is a chain and thus $(b \vee c)/\alpha$ is comparable to e/α, we can not be in case $\textcircled{\chi_2}$ and we have $L \cong \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_k$ for some $k \in [1, n-5]$, which has

$2^{n-4}+1$ congruences w.r.t. its unique weak complementation if $k>1$ and $2^{n-5}+1$ congruences w.r.t. any of its weak complementations if $k=1$, as noticed in Example 5.3. Therefore $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})|\leq 2^{n-4}+1$.
(xi) Assume that $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})|=2^{n-4}+1$, so that $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,^{\Delta[\alpha]})|\geq 2^{n-5}+1/2$ by Lemma 7.1 and thus $|\mathrm{Con}_{\mathbb{WCL}}(L/\alpha,^{\Delta[\alpha]})|\geq 2^{n-5}+1$.
Then, by the proof of (x), it follows that: $L\cong\mathcal{C}_{n-r-s+3}\oplus(\mathcal{C}_r\boxplus\mathcal{C}_s)$ for some $r,s\in\mathbb{N}\setminus\{0,1,2\}$ such that $r+s\leq n+2$ and $^{\Delta}$ is trivial if $r+s>6$ or $L\cong\mathcal{C}_{n-k-4}\oplus(\mathcal{C}_2\times\mathcal{C}_3)\oplus\mathcal{C}_k$ for some $k\in[2,n-5]$ or $L\cong\mathcal{C}_{n-r-s-4}\oplus\mathcal{C}_2^2\oplus\mathcal{C}_r\oplus\mathcal{C}_2^2\oplus\mathcal{C}_s$ or some $r,s\in\mathbb{N}^*$ such that $s>1$ and $r+s\leq n-5$.

Corollary 7.1 *For any $n\in\mathbb{N}^*$, any lattice L with $|L|=n$ and any weak dicomplementation $(^{\Delta},^{\nabla})$ on L, we have:*

1. $|\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})|\leq 2^{n-1}$;
2. $|\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})|=2^{n-1}$ *iff* $n\in\{1,2\}$;
3. $|\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})|=2^{n-2}$ *iff $n=4$ and* $\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})\cong\mathcal{C}_2^2$ *iff* $L\cong\mathcal{C}_2^2$ *and* $^{\Delta}=^{\nabla}$ *is the Boolean complementation*;
4. *if $L\ncong\mathcal{C}_2^2$ or its weak dicomplementation is not Boolean, then*: $|\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})|<2^{n-1}$ *iff* $|\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})|\leq 2^{n-3}+1$;
5. $|\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})|=2^{n-3}+1$ *iff* $\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})\cong\mathcal{C}_2^{n-3}\oplus\mathcal{C}_2$ *iff $n\geq 3$ and $L\cong\mathcal{C}_n$*;
6. *if* $|\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})|\leq 2^{n-3}$, *then* $|\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})|\leq 2^{n-4}+1$;
7. $|\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})|=2^{n-4}+1$ *iff* $\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})\cong\mathcal{C}_2^{n-4}\oplus\mathcal{C}_2$ *iff one of the following holds*: • $n\geq 5$, $L\cong\mathcal{C}_k\boxplus\mathcal{C}_{n-k+2}$ *for some $k\in[3,n-2]$ and $(^{\Delta},^{\nabla})$ is the trivial weak dicomplementation on L*; • $n\geq 6$ *and* $L\cong\mathcal{C}_k\oplus\mathcal{C}_2^2\oplus\mathcal{C}_{n-k-2}$ *for some* $k\in[2,n-4]$.

Proof Since $\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})=\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})\cap\mathrm{Con}_{\mathbb{DWCL}}(L,^{\nabla})$ and thus $|\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})|\leq|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})|,|\mathrm{Con}_{\mathbb{DWCL}}(L,^{\nabla})|$, it suffices to calculate the numbers of lattice congruences of the lattices in Theorem 7.2 that preserve the weak complementations in Theorem 7.2 and the dual weak complementations in the duals of the statements in this theorem. Actually, we only need to look at the lattices that appear both in these statements and their duals; for instance, if $n\geq 6$, then, by Theorem 7.2, (xi), $\mathcal{C}_{n-5}\oplus(\mathcal{C}_3\boxplus\mathcal{C}_5)$ can be organized as a weakly complemented lattice with $2^{n-4}+1$ congruences, but not as a dual weakly complemented lattice with at least $2^{n-4}+1$ congruences.

As in the proof of Theorem 7.2 for weak complementations, we use the observations at the beginning of Section 4 and Proposition 6.1 to determine the weak dicomplementations. We determine the congruence lattices w.r.t. these weak dicomplementations by using Lemma 4.1 along with Proposition 6.2 and Corollary 6.2.

$\mathcal{C}_n$ can only be endowed with the trivial weak dicomplementation, w.r.t. which $\mathcal{C}_1$ has $1=2^{1-1}$ congruences, $\mathcal{C}_2$ has $2=2^{2-1}$ congruences and, if $n\geq 3$, then $\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_n,^{\Delta\mathcal{C}_n},^{\nabla\mathcal{C}_n})\cong\mathrm{Con}(\mathcal{C}_{n-2})\oplus\mathcal{C}_2\cong\mathcal{C}_2^{n-3}\oplus\mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_n,^{\Delta\mathcal{C}_n},^{\nabla\mathcal{C}_n})|=2^{n-3}+1$.

We use the notations for weak complementations in the proof of Theorem 7.2 and we also denote, in the case when L has exactly two distinct atoms, by $^{\nabla L\setminus\{0\}}$ the unique nontrivial dual weak complementation on L, as in Proposition 5.1.

By Corollary 6.2, $\mathcal{C}_2^2$ has four weak dicomplementations, w.r.t. which: $|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_2^2,^{\Delta\mathcal{C}_2^2\setminus\{1\}},^{\nabla\mathcal{C}_2^2\setminus\{0\}})|=4=2^{|\mathcal{C}_2^2|-2}$, while $|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_2^2,^{\Delta\mathcal{C}_2^2},^{\nabla\mathcal{C}_2^2})|=|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_2^2,^{\Delta\mathcal{C}_2^2\setminus\{1\}},^{\nabla\mathcal{C}_2^2})|=|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_2^2,^{\Delta\mathcal{C}_2^2},^{\nabla\mathcal{C}_2^2\setminus\{0\}})|=2=2^{|\mathcal{C}_2^2|-3}$.

By Example 5.3, w.r.t. any of its 9 weak dicomplementations, $\mathcal{C}_2 \times \mathcal{C}_3$ has at most $4 = 2^{|\mathcal{C}_2 \times \mathcal{C}_3|-4}$ congruences.

If $n \geq 5$, then $\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2$ can only be endowed with two weak dicomplementations: the trivial one, $({}^{\Delta \mathcal{C}_{n-3}\oplus\mathcal{C}_2^2}, {}^{\nabla \mathcal{C}_{n-3}\oplus\mathcal{C}_2^2})$, and $({}^{\Delta(\mathcal{C}_{n-3}\oplus\mathcal{C}_2^2)\setminus\{1\}}, {}^{\nabla \mathcal{C}_{n-3}\oplus\mathcal{C}_2^2})$, w.r.t. which $\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2, {}^{\Delta \mathcal{C}_{n-3}\oplus\mathcal{C}_2^2}, {}^{\nabla \mathcal{C}_{n-3}\oplus\mathcal{C}_2^2}) = \mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2, {}^{\Delta(\mathcal{C}_{n-3}\oplus\mathcal{C}_2^2)\setminus\{1\}}, {}^{\nabla \mathcal{C}_{n-3}\oplus\mathcal{C}_2^2}) \cong \mathrm{Con}(\mathcal{C}_{n-4}) \oplus \mathcal{C}_2 \cong \mathcal{C}_2^{n-5} \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_{n-3} \oplus \mathcal{C}_2^2, {}^{\Delta \mathcal{C}_{n-3}\oplus\mathcal{C}_2^2}, {}^{\nabla \mathcal{C}_{n-3}\oplus\mathcal{C}_2^2})| = |\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_{n-3}\oplus \mathcal{C}_2^2, {}^{\Delta(\mathcal{C}_{n-3}\oplus\mathcal{C}_2^2)\setminus\{1\}}, {}^{\nabla \mathcal{C}_{n-3}\oplus\mathcal{C}_2^2})| = 2^{n-5} + 1$. Dually, if $n \geq 5$, then $\mathcal{C}_2^2 \oplus \mathcal{C}_{n-3}$ can only be endowed with two weak dicomplementations, w.r.t. which $|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_2^2 \oplus \mathcal{C}_{n-3}, {}^{\Delta \mathcal{C}_2^2\oplus\mathcal{C}_{n-3}}, {}^{\nabla \mathcal{C}_2^2\oplus\mathcal{C}_{n-3}})| = |\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_2^2 \oplus \mathcal{C}_{n-3}, {}^{\Delta \mathcal{C}_2^2\oplus\mathcal{C}_{n-3}}, {}^{\nabla(\mathcal{C}_2^2\oplus\mathcal{C}_{n-3})\setminus\{0\}})| = 2^{n-5} + 1$.

If $n \geq 6$, then, for any $k \in [2, n-4]$, $\mathcal{C}_k \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_{n-k-2}$ can only be endowed with the trivial weak dicomplementation, w.r.t. which $\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_k \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_{n-k-2}, {}^{\Delta \mathcal{C}_k\oplus\mathcal{C}_2^2\oplus\mathcal{C}_{n-k-2}}, {}^{\nabla \mathcal{C}_k\oplus\mathcal{C}_2^2\oplus\mathcal{C}_{n-k-2}}) \cong \mathrm{Con}(\mathcal{C}_{k-1} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_{n-k-3}) \oplus \mathcal{C}_2 \cong \mathcal{C}_{k-2+2+n-k-4} \oplus \mathcal{C}_2 = \mathcal{C}_2^{n-4} \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_k \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_{n-k-2}, {}^{\Delta \mathcal{C}_k\oplus\mathcal{C}_2^2\oplus\mathcal{C}_{n-k-2}}, {}^{\nabla \mathcal{C}_k\oplus\mathcal{C}_2^2\oplus\mathcal{C}_{n-k-2}})| = 2^{n-4} + 1$.

By analogous calculations to those in Example 6.1, if $n \geq 5$, then, for any $k \in [3, n-2]$, $\mathcal{C}_k \boxplus \mathcal{C}_{n-k+2}$ has four weak dicomplementations, w.r.t. which:

$|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_k \boxplus \mathcal{C}_{n-k+2}, {}^{\Delta \mathcal{C}_k\boxplus\mathcal{C}_{n-k+2}}, {}^{\nabla \mathcal{C}_k\boxplus\mathcal{C}_{n-k+2}})| = 2^{n+2-6} + 1 = 2^{n-4} + 1$;

$|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_k \boxplus \mathcal{C}_{n-k+2}, {}^{\Delta(\mathcal{C}_k\boxplus\mathcal{C}_{n-k+2})\setminus\{1\}}, {}^{\nabla \mathcal{C}_k\boxplus\mathcal{C}_{n-k+2}})| = |\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_k \boxplus \mathcal{C}_{n-k+2}, {}^{\Delta \mathcal{C}_k\boxplus\mathcal{C}_{n-k+2}}, {}^{\nabla(\mathcal{C}_k\boxplus\mathcal{C}_{n-k+2})\setminus\{0\}})| = \begin{cases} 2^{n-k+2-4} + 1 = 2^{n-5} + 1, & k = 3, \\ 2^{n-6} + 1, & k \geq 4; \end{cases}$

$|\mathrm{Con}_{\mathbb{WDL}}(\mathcal{C}_k \boxplus \mathcal{C}_{n-k+2}, {}^{\Delta(\mathcal{C}_k\boxplus\mathcal{C}_{n-k+2})\setminus\{1\}}, {}^{\nabla(\mathcal{C}_k\boxplus\mathcal{C}_{n-k+2})\setminus\{0\}})| =$

$$\begin{cases} 2 \leq 2^{n-4}, & k \leq 4, n-k \leq 2; \\ 2^{n-k-3} + 1 \leq 2^{n-6} + 1, & k \leq 4, n-k \geq 3; \\ 2^{n-8} + 1, & k \geq 5, n-k \geq 3. \end{cases}$$

If $n \geq 8$ and $L \cong \mathcal{C}_k \oplus (\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_{n-k-4}$ for some $k \in [2, n-6]$ or $n \geq 9$ and $L \cong \mathcal{C}_{n-r-s-4} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_r \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_s$ for some $r, s \in \mathbb{N}^*$ such that $s > 1$ and $r + s \leq n - 6$, then L can only be endowed with the trivial weak dicomplementation and $\mathrm{Con}_{\mathbb{WDL}}(L, {}^{\Delta L}, {}^{\nabla L}) \cong \mathcal{C}_2^{n-5} \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{WDL}}(L, {}^{\Delta L}, {}^{\nabla L})| = 2^{n-5} + 1$.

8 Conclusions

After a preliminary investigation on the (dual) weak complementations that can be defined on bounded lattices with different structures, we determine the several largest numbers of congruences of the n–element (dual) weakly complemented lattices and those of the n–element weakly dicomplemented lattices, for n an arbitrary nonzero natural number. In these n–element algebras having one of these largest numbers of congruences, the (dual) weak complementations are not uniquely determined by their lattice structures, but they are all representable, hence the weakly dicomplemented lattices in this list (actually all bounded lattices in Theorem 7.2 endowed with any pair of one of these weak complementations with a representable dual weak complementation, and dually) are canonical concept algebras associated to contexts determined by pairs of a join–dense subset and a meet–dense subset of their underlying bounded lattices; this sheds more light on the representability problem for weak dicomplementations on finite lattices, which could be worth investigating with some of the tools we have developed in this paper.

Acknowledgements This work was supported by the research grant number IZSEZO_186586/1, awarded to the project *Reticulations of Concept Algebras* by the Swiss National Science Foundation, within the programme Scientific Exchanges. We thank the anonymous referee for his or her good suggestions for improving the form of our paper.

Funding Open access funding provided by Bern University of Applied Sciences

References

1. Bishop, A.A.: Completions of Lattices with Semicomplementation. Dissertation, Western Michigan University (1972)
2. Czédli, G., Mureşan, C.: On Principal Congruences and the Number of Congruences of a Lattice with More Ideals than Filters. Acta Universitatis Szegediensis, Acta Scientiarum Mathematicarum (2019)
3. Czédli, G.: A Note on Finite Lattices with Many Congruences, Acta Universitatis Matthiae Belii. Series Mathematics Online , 22–28 (2018)
4. Czédli, G.: Finite Semilattices with Many Congruences. Order **36**(2), 233–247 (2019)
5. Freese, R.: Computing Congruence Lattices of Finite Lattices. Proc. Amer. Math. Soc. **125**, 3457–3463 (1997)
6. Ganter, B., Wille, R.: Formal Concept Analysis. Springer, Mathematical Foundations (1999)
7. Ganter, B., Kwuida, L.: Finite Distributive Concept Algebras. Order **23**, 235–248 (2006)
8. Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., Scott, D.S.: Continuous lattices and domains, Encyclopedia of Mathematics and its Applications, vol. 93, pp xxxvi+591. Cambridge University Press, Cambridge (2003)
9. Giuntini, R., Mureşan, C., Paoli, F., PBZ* –lattices: Ordinal and Horizontal Sums, Algebraic Perspectives on Substructural Logics, Trends in Logic Series, vol. 55, pp. 73–105. Springer Nature (2020)
10. Gorbunov, V.A.: Algebraic Theory of Quasivarieties (translated from Russian), Siberian School of Algebra and Logic, pp. xii+298. Consultants Bureau, New York (1998)
11. Grätzer, G.: General Lattice Theory. Birkhäuser Akademie-Verlag, Basel-Boston-Berlin (1978)
12. Grätzer, G.: Universal Algebra, 2nd edn. Springer Science+Business Media, LLC, New York (2008)
13. Grätzer, G.: The Congruences of Finite Lattices. A "proof–by–picture" Approach, Second ed.. Birkhäuser–Springer, Cham (2016)
14. Kwuida, L.: Congruences of Concept Algebras, Proceedings of Novi Sad Algebraic Conference 2003. Novi. Sad. J. Math. **34**(2), 141–152 (2004)
15. Kwuida, L.: Dicomplemented Lattices. A Contextual Generalization of Boolean Algebras. Dissertation, TU Dresden, Shaker–Verlag (2004)
16. Kwuida, L., Machida, H.: On the Isomorphism Problem of Weakly Dicomplemented Lattices. Ann. Math. Artif. Intell. **59**(2), 223–239 (2010)
17. Mureşan, C.: On the Cardinalities of the Sets of Congruences, Ideals and Filters of a Lattice, Analele Universităţii Bucureşti. Seria Informatică. Proceedings of the Workshop Days of Computer Science (DACS): LXII, affiliated workshop of the 11th edition of the conference Computability in Europe (2015), 55–68. University of Bucharest, Bucharest, Romania (2015)
18. Mureşan, C.: Cancelling Congruences of Lattices, While Keeping Their Numbers of Filters and Ideals, South American Journal of Logic (2020)
19. Mureşan, C., Kulin, J.: On the Largest Numbers of Congruences of Finite Lattices. Order **37**(3), 445–460 (2020)
20. Mureşan, C.: Some Properties of Lattice Congruences Preserving Involutions and Their Largest Numbers in the Finite Case, Houston Journal of Mathematics **47**(2), 295–320 (2021)
21. Mureşan, C.: A Note on Congruences of Infinite Bounded Involution Lattices. Scientific Annals of Computer Science **31**(1), 51–78 (2021)

22. Szász, G.: On Weakly Complemented Lattices. Acta Sci. Math. (Szeged) **16**(1–2), 122–126 (1955)
23. Trullenque Ortiz, C.: Complete Theories of Boolean Algebras. Grade Thesis, University of Barcelona, Faculty of Mathematics and Computer Science (2018)
24. Wille, R.: Boolean Concept Logic. In Ganter, B., Mineau, G.W. (eds.) ICCS 2000, Conceptual Structures: Logical, Linguistic, and Computational Issues, vol. 1867, pp. 317–331. Springer LNAI (2000)